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Fluid relabelling symmetries, Lie point symmetries and the Lagrangian map in magnetohydrodynamics and gas dynamics

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Abstract
We explore the role of the Lagrangian map for Lie symmetries in magnetohydrodynamics (MHD) and gas dynamics. By converting the Eulerian Lie point symmetries of the Galilei group to Lagrange label space, in which the Eulerian position coordinate $\mathbf{x}$ is regarded as a function of the Lagrange fluid labels $\mathbf{x}_0$ and time t, one finds that there is an infinite class of symmetries in Lagrange label space that map onto each Eulerian Lie point symmetry of the Galilei group. The allowed transformation of the Lagrangian fluid labels $\mathbf{x}_0$ corresponds to a fluid relabelling symmetry, including the case where there is no change in the fluid labels. We also consider a class of three, well-known, scaling symmetries for a gas with a constant adiabatic index γ. These symmetries map onto a modified form of the fluid relabelling symmetry determining equations, with non-zero source terms. We determine under which conditions these symmetries are variational or divergence symmetries of the action, and determine the corresponding Lagrangian and Eulerian conservation laws by use of Noether's theorem. These conservation laws depend on the initial entropy, density and magnetic field of the fluid. We derive the conservation law corresponding to the projective symmetry in gas dynamics, for the case $\gamma = (n+2)/n$, where n is the number of Cartesian space coordinates, and the corresponding result for two-dimensional (2D) MHD, for the case $\gamma = 2$. Lie algebraic structures in Lagrange label space corresponding to the symmetries are investigated. The Lie algebraic symmetry relations between the fluid relabelling symmetries in Lagrange label space, and their commutators with a linear combination of the three symmetries with a constant adiabatic index are delineated.

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1. Introduction

There is an extensive literature on the symmetries and Hamiltonian structure of the ideal gas dynamic equations (e.g. Salmon (1982, 1988), Ibragimov (1994), Nutku (1987), Olver and Nutku (1988), Morrison (1998), Holm *et al* (1998), Hydon (2005), Bridges *et al* (2005), Marsden and Ratiu (1994)) and the magnetohydrodynamic equations (e.g. Morrison (1982), Fuchs (1991), Padhye and Morrison (1996a, 1996b), Padhye (1998), Holm *et al* (1998), Kuznetsov and Ruban (2000)). The Lie point symmetry algebra of the ideal, compressible gas dynamic and MHD equations have been obtained by Fuchs (1991). The classification of the Lie algebra and sub-algebras of these equations has been carried out by Grundland and Lalague (1995). The Lie point symmetries of the equations obtained by Fuchs (1991) pertain to the Eulerian form of the equations.

The MHD equations and gas dynamic systems admit the ten-parameter Galilei Lie group. This includes the space and time translation symmetries, the space rotations and the Galilean boosts. This group has the Lie algebra basis of vector fields:

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = \frac{\partial}{\partial z}, \qquad X_4 = \frac{\partial}{\partial t}, \tag{1.1}$$

$$X_5 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u^x}, \qquad X_6 = t\frac{\partial}{\partial y} + \frac{\partial}{\partial u^y}, \qquad X_7 = t\frac{\partial}{\partial z} + \frac{\partial}{\partial u^z}, \tag{1.2}$$

$$X_8 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + u^z\frac{\partial}{\partial u^y} - u^y\frac{\partial}{\partial u^z} + B^z\frac{\partial}{\partial B^y} - B^y\frac{\partial}{\partial B^z}, \tag{1.3}$$

$$X_9 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} + u^x\frac{\partial}{\partial u^z} - u^z\frac{\partial}{\partial u^x} + B^x\frac{\partial}{\partial B^z} - B^z\frac{\partial}{\partial B^x}, \tag{1.4}$$

$$X_{10} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + u^y\frac{\partial}{\partial u^x} - u^x\frac{\partial}{\partial u^y} + B^y\frac{\partial}{\partial B^x} - B^x\frac{\partial}{\partial B^y}. \tag{1.5}$$

In the above equations (t, x, y, z) refer to the time and rectangular Cartesian space coordinates, $\mathbf{u}$ is the fluid velocity; $\mathbf{B}$ is the magnetic field induction. We use ρ and p to denote the gas density and pressure. The Lie symmetry operators $\{X_1, X_2, X_3\}$ represent the space translation symmetries, and correspond via Noether's theorem to the momentum conservation equations along the x, y and z axes respectively; X_4 is the time translation symmetry and corresponds to the energy conservation equation; $\{X_5, X_6, X_7\}$ correspond to the Galilean boosts and give rise to the uniform center-of-mass conservation laws; $\{X_8, X_9, X_{10}\}$ correspond to rotational invariance about the x, y and z axes respectively and give rise to the angular momentum laws. These conservation laws are derived by Morrison (1982) using a non-canonical Poisson bracket formalism. They are also derived by Padhye (1998) and Webb *et al* (2005b) using a Lagrangian form of the MHD action principle and Noether's first theorem.

In addition to the above symmetries, there is a class of infinite-dimensional fluid relabelling symmetries, that leave the MHD action invariant under transformation of the Lagrangian fluid labels. The conservation laws in this latter case are associated with Noether's second theorem (e.g. Padhye (1998), Padhye and Morrison (1996a, 1996b), Webb *et al* (2005b)). Ertel's theorem for the conservation of potential vorticity is a consequence of an infinite class of fluid relabelling symmetries and Noether's second theorem. The fluid relabelling symmetries are obtained by searching for Lie transformations that leave the Lagrangian action invariant and involve only transformation of the Lagrangian fluid labels $\mathbf{x}_0$ (e.g. Salmon (1982, 1988), Padhye and Morrison (1996a, 1996b), Padhye (1998), Webb *et al* (2005b)). In general one can search for Lie transformations of the form

$$\mathbf{x}' = \mathbf{x} + \epsilon V^{\mathbf{x}}, \qquad t' = t + \epsilon V^t, \qquad \mathbf{x}_0' = \mathbf{x}_0 + \epsilon V^{\mathbf{x}_0} \tag{1.6}$$

that leave the action invariant, up to a divergence transformation, where $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)$ is the Lagrangian map between the Eulerian fluid particle position and its Lagrangian label $\mathbf{x}_0$. The Lagrangian map is the solution of the differential equation system $\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbf{u}(\mathbf{x}, t)$, for $\mathbf{x}$, where $\mathbf{u}$ is the fluid velocity, subject to the initial condition $\mathbf{x} = \mathbf{x}_0$ at time $t = 0$. The fluid relabelling symmetries correspond to variational symmetries of the action of the form $\mathbf{x}' = \mathbf{x}$, $t' = t$ and $\mathbf{x}_0' = \mathbf{x}_0 + \epsilon V^{\mathbf{x}_0}$, in which $\mathbf{x}$ and t are fixed.

The allowed Lagrangian maps $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)$ are the solutions of the Euler–Lagrange equations for the Lagrangian form of the MHD action. These equations are equivalent to the Lagrangian momentum equation for the MHD fluid, and consist of three nonlinear, coupled wave equations for $\mathbf{x}(\mathbf{x}_0, t)$, in which the Lagrangian density $\rho_0(\mathbf{x}_0)$, entropy $S(\mathbf{x}_0)$ and magnetic induction $\mathbf{B}_0(\mathbf{x}_0)$ are given functions of $\mathbf{x}_0$. In other words, the partial differential equation (PDE) systems consisting of (i) the Eulerian MHD equations and (ii) the coupled, nonlinear, Lagrangian wave equations for $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)$, for given $\rho_0(\mathbf{x}_0)$, $S(\mathbf{x}_0)$ and $\mathbf{B}_0(\mathbf{x}_0)$ are equivalent systems via the Lagrangian map (see (2.19) *et seq* and appendix A)

A question we address is how are the Eulerian Lie symmetries, as exemplified by the Galilei group symmetry operators (1.1)–(1.5) related to the Lagrangian symmetry transformations (1.6)? One answer to this question has been provided by Padhye and Morrison (1996a, 1996b) and Padhye (1998) who characterized the Galilei group by transformations using the Lagrangian map in which $V^{\mathbf{x}_0} = 0$ but with $V^{\mathbf{x}} \neq 0$ and $V^t \neq 0$. We explore Lie transformations for which all of the infinitesimal Lie generators V^t, $V^{\mathbf{x}}$ and $V^{\mathbf{x}_0}$ are non-zero. We use the Lie extension formulae for transformation of derivatives of the dependent variables (e.g. Ibragimov (1985), Bluman and Kumei (1989), Olver (1993)), to convert Lie symmetries from the Eulerian form to their corresponding form in Lagrange label space. We find that the Lie point symmetries of the Galilei group (1.1)–(1.5) correspond to an infinite class of Lie symmetries of the form (1.6) in Lagrange label space, in which $V^{\mathbf{x}_0}$ satisfy the Lie determining equations for the fluid relabelling symmetries. The equations also admit solutions with $V^{\mathbf{x}_0} = 0$, which correspond to the Lagrangian description of the Lie point symmetries of the Galilei group in Padhye and Morrison (1996a, 1996b) and Padhye (1998).

The Galilei symmetries do not depend on the equation of state of the gas. However, there is a class of Lie point symmetries of the Eulerian gas dynamic and MHD equations that apply if the gas has a polytropic equation of state of the form

$$\varepsilon = \frac{p}{\gamma - 1}, \qquad S = C_v \ln\left[\frac{p}{p_1}\left(\frac{\rho_1}{\rho}\right)^{\gamma}\right], \qquad p = p_1\left(\frac{\rho}{\rho_1}\right)^{\gamma} \exp(\bar{S}), \tag{1.7}$$

where ρ_1 and p_1 are constant normalizing values of the density and gas pressure, ε is the internal energy density of the gas, S is the entropy ($\bar{S} = S/C_v$), C_v is the specific heat of the gas at constant volume and γ is the adiabatic index of the gas. Using $(t, x, y, z, u^x, u^y, u^z, B^x, B^y, B^z, p, \rho)^t$ as variables in the Eulerian MHD equations, Fuchs (1991) obtained the scaling symmetries:

$$X_{11} = t\frac{\partial}{\partial t} - u^x\frac{\partial}{\partial u^x} - u^y\frac{\partial}{\partial u^y} - u^z\frac{\partial}{\partial u^z} + 2\rho\frac{\partial}{\partial \rho}, \tag{1.8}$$

$$X_{12} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + u^x\frac{\partial}{\partial u^x} + u^y\frac{\partial}{\partial u^y} + u^z\frac{\partial}{\partial u^z} - 2\rho\frac{\partial}{\partial \rho}, \tag{1.9}$$

$$X_{13} = B^x\frac{\partial}{\partial B^x} + B^y\frac{\partial}{\partial B^y} + B^z\frac{\partial}{\partial B^z} + 2\rho\frac{\partial}{\partial \rho} + 2p\frac{\partial}{\partial p}. \tag{1.10}$$

If the adiabatic index of the gas $\gamma = (n+2)/n$, where n is the number of Cartesian space dimensions ($\gamma = 5/3, 2, 3$ for $n = 3, 2, 1$ respectively), then the ideal gas dynamic equations admit the projective symmetry (e.g. Ovsjannikov (1962), Ibragimov (1985)):

$$X_{14} = tx^\alpha \frac{\partial}{\partial x^\alpha} + (x^i - tu^i)\frac{\partial}{\partial u^i} - nt\rho\frac{\partial}{\partial \rho} - (n+2)tp\frac{\partial}{\partial p}. \tag{1.11}$$

In (1.11) $x^\alpha = (t, x, y, z)^t$, and we use the Einstein summation convention for repeated indices. The Greek indices $\alpha = 0, 1, 2, 3$ correspond to the spacetime coordinates (t, x, y, z), and the Latin indices $i = 1, 2, 3$ pertain to the space coordinates (x, y, z).

The Lie algebra of the symmetries $\{X_j : 1 \leqslant j \leqslant 14\}$ and the classification of the subgroups and conjugacy classes of the Lie algebra are given in Grundland and Lalague (1995). Grundland and Lalague use the notation

$$\begin{aligned} &P_\mu = \frac{\partial}{\partial x^\mu}, \qquad K_i = x^0\frac{\partial}{\partial x^i} + \frac{\partial}{\partial u^i}, \\ &J_k = \epsilon_{kij}\left(x^j\frac{\partial}{\partial x^i} + u^j\frac{\partial}{\partial u^i} + B^j\frac{\partial}{\partial B^i}\right) \end{aligned} \tag{1.12}$$

to describe the Galilei group (1.1)–(1.5). The symmetries P_μ ($\mu = 0, 1, 2, 3$) correspond to the time and space translation symmetries (1.1), where $x^\mu = (t, x, y, z)$. K_i ($i = 1, 2, 3$) are the Galilean boosts (1.2) and J_k are the rotational symmetries (1.3)–(1.5). ϵ_{ijk} is the Levi-Civita symbol or antisymmetric tensor density. Grundland and Lalague use the symmetries: $F = X_{11} + X_{12}$, $G = X_{13} - X_{11}$ and $H = X_{13}$ as alternative basis vector fields instead of X_{11}, X_{12} and X_{13}, and use the symbol $C \equiv X_{14}$ for the projective symmetry X_{14}. A more complete list of the various forms of the gas dynamic equations in one, two and three space dimensions, and related equations, such as the Monge–Ampere equations is given in Ibragimov (1994).

We address the question: what are the conservation laws (if any) associated with the symmetries (1.8)–(1.11) using Noether's theorems? We use the form of Noether's theorem based on the Lagrangian action principle developed by Newcomb (1962). Alternative variational principles using Clebsch potentials to incorporate the constraints of mass conservation, entropy advection, and the frozen magnetic field (Faraday's law) could also in principle be used (e.g. Lundgren (1963), Holm and Kupershmidt (1983a, 1983b), Zakharov and Kuznetsov (1984)).

If the generalized symmetry operator

$$X_{(a)} = \alpha X_{11} + \beta X_{12} + \delta X_{13} \tag{1.13}$$

is a divergence symmetry or variational symmetry of the action, then Noether' first theorem can be used to write down the corresponding conservation law (e.g. Olver (1993), Bluman and Kumei (1989)). We use the subscript a in (1.13) to emphasize that this symmetry is associated with an adiabatic equation of state for the gas, with a constant adiabatic index γ. It turns out that the condition for the symmetry (1.13) to be a variational symmetry of the action is that

$$\alpha + 2\delta + n\beta = 0, \tag{1.14}$$

where n is the number of Lagrangian fluid labels $\mathbf{x}_0$ in the variational principle, and α, β and δ are constants.

In section 2 we present the Lagrangian variational principle for MHD and the Lagrangian map developed by Newcomb (1962). Section 3 derives Noether's theorem for Lagrangian MHD and the condition that must be satisfied for a divergence symmetry of the action. Both the Lagrangian and Eulerian form of the conservation laws for a given Lie symmetry are obtained. The Lie determining equations for a fluid relabelling symmetry are discussed.

The Eulerian Lie point symmetries of the Galilei group and the projective symmetry, for $\gamma = (n+2)/n$ of (1.11) in gas dynamics, are converted to the Lagrangian form by using the Lie extension formulae for the transformation of derivatives of the dependent variables. Section 4 concerns the determination of the Lagrangian form of the Eulerian point symmetry $X_{(a)}$ of (1.13) for an adiabatic gas with a constant adiabatic index γ, and condition (1.14) for the symmetry to be a variational symmetry of the action is derived. Noether's theorem is used in section 5 to obtain the MHD conservation law for the symmetry $X_{(a)}$ for the case when $\alpha + 2\delta + n\beta = 0$. Section 6 also obtains conservation laws for the symmetry $X_{(a)}$ for gas dynamics when there is no magnetic field present, as well as the conservation law associated with the projective symmetry X_{14} in (1.11) (see also appendix C and Fuchs and Richter (1987) for the case of 2D MHD, with an ignorable coordinate z, with $\mathbf{B} = (0, 0, B)^t$). Section 7 considers the Lie algebra of vector fields in Lagrange label space for both the relabelling symmetries and the generalized symmetry $X_{(a)}$. The Lie brackets of the fluid relabelling symmetries with the Galilei, Lie point symmetries with $V^{\mathbf{x}_0} = 0$ are all zero. Section 8 concludes with a summary and discussion.

2. Magnetohydrodynamics

In this section we consider the variational form of the MHD equations, using the Lagrangian, variational approach of Newcomb (1962).

The time-dependent Eulerian MHD equations consist of the mass, momentum and entropy advection equations

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{2.1}$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p + \mathbf{J}\times\mathbf{B} - \rho\nabla\Phi, \tag{2.2}$$

$$\frac{\partial S}{\partial t} + \mathbf{u}\cdot\nabla S = 0, \tag{2.3}$$

coupled with Maxwell's equations in the MHD limit:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B}), \tag{2.4}$$

$$\mathbf{J} = \frac{\nabla\times\mathbf{B}}{\mu_0}, \qquad \nabla\cdot\mathbf{B} = 0. \tag{2.5}$$

Maxwell's equations (2.4) and (2.5) correspond to Faraday's induction equation, Ampere's law for the current $\mathbf{J}$ and Gauss's equation $\nabla\cdot\mathbf{B} = 0$. The force $\mathbf{F}_g = -\rho\nabla\Phi$ is the force associated with the external gravitational potential Φ. The above equations need to be supplemented by an equation of state for the gas internal energy density $\varepsilon = \varepsilon(\rho, S)$ and the second law of thermodynamics. For an ideal gas, the second law of thermodynamics, for an ideal gas, gives

$$p = \rho\frac{\partial\varepsilon}{\partial\rho} - \varepsilon, \qquad \rho T = \frac{\partial\varepsilon}{\partial S}, \tag{2.6}$$

for the pressure p and the temperature T of the gas.

2.1. The Lagrangian map and variational formulation

The Lagrangian map: $\mathbf{x} = \mathbf{X}(\mathbf{x}_0, t)$ is obtained by integrating the fluid velocity equation $\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbf{u}(\mathbf{x}, t)$, subject to the initial condition $\mathbf{x} = \mathbf{x}_0$ at time $t = 0$. In this approach, the mass continuity equation and entropy advection equation are replaced by the equivalent algebraic equations

$$\rho = \frac{\rho_0(\mathbf{x}_0)}{J}, \qquad S = S(\mathbf{x}_0), \tag{2.7}$$

where

$$J = \det(x_{ij}) \qquad \text{and} \qquad x_{ij} = \frac{\partial x^i(\mathbf{x}_0, t)}{\partial x_0^j}. \tag{2.8}$$

Similarly, Faraday's equation (2.4) has the formal solution for the magnetic field induction $\mathbf{B}$ of the form

$$B^i = \frac{x_{ij} B_0^j}{J}, \qquad \nabla_0 \cdot \mathbf{B}_0 = 0. \tag{2.9}$$

Solution (2.9) for B^i is equivalent to the frozen in field theorem in MHD (e.g. Stern (1966), Parker (1979)), and the initial condition $\nabla_0 \cdot \mathbf{B}_0 = 0$ is imposed in order to ensure that Gauss's law $\nabla \cdot \mathbf{B} = 0$ is satisfied.

The Lagrangian map $\mathbf{x} = \mathbf{X}(\mathbf{x}_0, t)$ and its inverse $\mathbf{x}_0 = \mathbf{X}_0(\mathbf{x}, t)$ are characterized by the relations

$$x_{is} y_{sp} = \delta^i_p, \qquad \frac{\partial x^i}{\partial t} + x_{is} \frac{\partial x_0^s}{\partial t} = 0, \tag{2.10}$$

where

$$x_{is} = \frac{\partial x^i}{\partial x_0^s} \qquad \text{and} \qquad y_{sp} = \frac{\partial x_0^s}{\partial x^p}. \tag{2.11}$$

From (2.10) and (2.11) we obtain

$$\frac{\partial x_0^i}{\partial t} + u^s \frac{\partial x_0^i}{\partial x^s} = 0, \tag{2.12}$$

showing that the Lagrange label $\mathbf{x}_0$ is advected with the background flow with velocity $\mathbf{u} = \partial \mathbf{x}(\mathbf{x}_0, t)/\partial t$.

From Cramer's rule

$$y_{ij} = \frac{A_{ji}}{J}, \qquad x_{ij} = J B_{ji}, \tag{2.13}$$

where $A_{ij} = \mathrm{cofac}(x_{ij})$ and $B_{ij} = \mathrm{cofac}(y_{ij})$ are the co-factor matrices associated with x_{ij} and y_{ij} (note A_{ij} and B_{ij} are inverse matrices). One can show:

$$A_{ij} = \tfrac{1}{2}\epsilon_{ipq}\epsilon_{jmn} x_{pm} x_{qn}, \qquad B_{ij} = \tfrac{1}{2}\epsilon_{ipq}\epsilon_{jmn} y_{pm} y_{qn}, \tag{2.14}$$

where ϵ_{ijk} is the anti-symmetric permutation tensor density (see, e.g., Newcomb (1962)). From (2.14) it follows that $\partial A_{ij}/\partial x_0^j = 0$ and $\partial B_{ij}/\partial x^j = 0$.

The action for the MHD system is

$$A = \iint \mathcal{L} d^3x \, \mathrm{d}t \equiv \iint \mathcal{L}^0 d^3x_0 \, \mathrm{d}t, \tag{2.15}$$

where

$$\mathcal{L} = \frac{1}{2}\rho|\mathbf{u}|^2 - \varepsilon(\rho, S) - \frac{B^2}{2\mu} - \rho\Phi, \qquad \mathcal{L}^0 = \mathcal{L}J, \tag{2.16}$$

are the Eulerian ($\mathcal{L}$) and Lagrangian ($\mathcal{L}^0$) Lagrange densities respectively. Using (2.7)–(2.9), and (2.16) we obtain

$$\mathcal{L}^0 = \frac{1}{2}\rho_0|\mathbf{x}_t|^2 - J\varepsilon\left(\frac{\rho_0}{J}, S\right) - \frac{x_{ij}x_{is}B_0^j B_0^s}{2\mu J} - \rho_0\Phi, \tag{2.17}$$

for $\mathcal{L}^0$. Note that in the Lagrange density $\mathcal{L}^0 = \mathcal{L}^0(\mathbf{x}_0, t; \mathbf{x}, \mathbf{x}_t, x_{ij})$, $\mathbf{x}_0$ and t are the independent variables, and $\mathbf{x}$ and its derivatives with respect to $\mathbf{x}_0$ and t are dependent variables.

Extremization of the action in (2.15) gives the Euler–Lagrange equations:

$$\frac{\delta A}{\delta x^i} = \frac{\partial \mathcal{L}^0}{\partial x^i} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}^0}{\partial x_t^i}\right) - \frac{\partial}{\partial x_0^s}\left(\frac{\partial \mathcal{L}^0}{\partial x_{ij}}\right) = 0, \tag{2.18}$$

where $x_{ij} \equiv \partial x^i/\partial x_0^j$. Evaluation of the variational derivative (2.18) gives the Lagrangian momentum equation for the system in the form (Newcomb (1962))

$$\rho_0\left(\frac{\partial^2 x^i}{\partial t^2} + \frac{\partial \Phi}{\partial x^i}\right) + \frac{\partial}{\partial x_0^j}\left\{A_{kj}\left[\left(p + \frac{B^2}{2\mu}\right)\delta^{ik} - \frac{B^i B^k}{\mu_0}\right]\right\} = 0, \tag{2.19}$$

where $A_{kj} = \mathrm{cofac}(x_{kj})$. Dividing (2.19) by J, and using the fact that $\partial A_{kj}/\partial x_0^j = 0$, gives the Eulerian form of the momentum equation (2.2).

Note that (2.19) can be reduced to three coupled nonlinear wave equations for $x^i = x^i(\mathbf{x}_0, t)$ for an equation of state $p = p(\rho, S)$ for the gas, in which $\rho_0(\mathbf{x}_0)$, $S(\mathbf{x}_0)$ and $\mathbf{B}_0(\mathbf{x}_0)$ are given functions of $\mathbf{x}_0$ where $\nabla_0 \cdot \mathbf{B}_0 = 0$ (see appendix A). The characteristic manifolds of (2.19) correspond to the usual Alfvén, fast and slow magnetoacoustic waves respectively (appendix A). Thus, the allowed Lagrangian maps $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)$ are solutions of the wave equations (A.1) with $\mathbf{x} = \mathbf{x}_0$ at time $t = 0$.

The above analysis uses Lagrangian variations of the action in which $\mathbf{x}_0$ is fixed. The Lagrangian variation of $\mathbf{x}(\mathbf{x}_0, t; \epsilon)$ is defined as $\Delta\mathbf{x} = \partial\mathbf{x}/\partial\epsilon$ evaluated at $\epsilon = 0$ and keeping $\mathbf{x}_0$ fixed. It is also possible to extremize the first form of the action in (2.15) using Eulerian variations in which $\mathbf{x}$ is held constant, leading to the Eulerian form of the momentum equation (2.2).

3. Symmetries and Noether's theorem in MHD

In this section we discuss Noether's first theorem in MHD. The analysis is similar to that in Padhye (1998) and Webb *et al* (2005b). We consider the Lagrangian form of action (2.15), namely

$$A = \iint \mathcal{L}^0 d^3x_0\, \mathrm{d}t, \tag{3.1}$$

where the Lagrangian density $\mathcal{L}^0$ is given by (2.17).

Proposition 3.1 (Noether's theorem)**.** *If action (3.1) is invariant to $O(\epsilon)$ under the infinitesimal Lie transformations:*

$$x'^i = x^i + \epsilon V^{x^i}, \qquad x_0'^j = x_0^j + \epsilon V^{x_0^j}, \qquad t' = t + \epsilon V^t, \tag{3.2}$$

and the divergence transformation:

$$\mathcal{L}^{0'} = \mathcal{L}^0 + \epsilon D_\alpha \Lambda_0^\alpha + O(\epsilon^2), \tag{3.3}$$

(here $D_0 \equiv \partial/\partial t$ and $D_i \equiv \partial/\partial x_0^i$ are the total derivative operators in the jet space consisting of the derivatives of $x^k(\mathbf{x}_0, t)$ and physical quantities that depend on $\mathbf{x}_0$ and t) then the MHD system admits the Lagrangian conservation law:

$$\frac{\partial I^0}{\partial t} + \frac{\partial I^j}{\partial x_0^j} = 0, \tag{3.4}$$

where

$$I^0 = \rho_0 u^k \hat{V}^{x^k} + V^t \mathcal{L}^0 + \Lambda_0^0, \tag{3.5}$$

$$I^j = \hat{V}^{x^k} \left[\left(p + \frac{B^2}{2\mu} \right) \delta^{ks} - \frac{B^k B^s}{\mu} \right] A_{sj} + V^{x_0^j} \mathcal{L}^0 + \Lambda_0^j, \tag{3.6}$$

In (3.5) and (3.6)

$$\hat{V}^{x^k(\mathbf{x}_0,t)} = V^{x^k(\mathbf{x}_0,t)} - \left(V^t \frac{\partial}{\partial t} + V^{x_0^s} \frac{\partial}{\partial x_0^s} \right) x^k(\mathbf{x}_0, t), \tag{3.7}$$

is the canonical Lie symmetry transformation generator corresponding to the Lie transformation (3.2) (i.e. $x'^k = x^k + \epsilon \hat{V}^{x^k}$, $t' = t$, $x_0'^j = x_0^j$).

Proof. Using Noether's theorem (e.g. Bluman and Kumei (1989)) we obtain

$$\begin{aligned} I^0 &= W^0 + V^t \mathcal{L}^0 + \Lambda_0^0 \equiv \frac{\partial \mathcal{L}^0}{\partial x_t^k} \hat{V}^{x^k} + V^t \mathcal{L}^0 + \Lambda_0^0, \\ I^j &= W^j + \mathcal{L}^0 V^{x_0^j} + \Lambda_0^j \equiv \frac{\partial \mathcal{L}^0}{\partial x_{kj}} \hat{V}^{x^k} + \mathcal{L}^0 V^{x_0^j} + \Lambda_0^j, \end{aligned} \tag{3.8}$$

for the conserved density I^0 and flux components I^j. Using (2.17) for $\mathcal{L}^0$ in (3.8) to evaluate the derivatives of $\mathcal{L}^0$ with respect to x_t^k and x_{ij} gives expressions (3.5) and (3.6) for I^0 and I^j. Proofs of Noether's first theorem are given in Bluman and Kumei (1989) and Olver (1993) (see Webb *et al* (2005b) for Noether's theorem for the MHD system, including the effects of fully nonlinear waves). □

Comment 1

The condition for action (3.1) to be invariant to $O(\epsilon)$ under the divergence transformation of the form (3.2) and (3.3) is

$$\tilde{X}\mathcal{L}^0 + \mathcal{L}^0 \left(D_t V^t + D_{x_0^j} V^{x_0^j} \right) + D_t \Lambda_0^0 + D_{x_0^j} \Lambda_0^j = 0, \tag{3.9}$$

where

$$\tilde{X} = V^t \frac{\partial}{\partial t} + V^{x^k} \frac{\partial}{\partial x^k} + V^{x_t^k} \frac{\partial}{\partial x_t^k} + V^{x_{kj}} \frac{\partial}{\partial x_{kj}} + \cdots, \tag{3.10}$$

is the extended Lie transformation operator acting on the jet space of the Lie transformation (3.2). Note that $\tilde{X}$ gives the rules for transforming derivatives of $x^k(\mathbf{x}_0, t)$ under Lie transformation (3.2). From Ibragimov (1985)

$$\tilde{X} = \hat{X} + V^\alpha D_\alpha, \tag{3.11}$$

$$\hat{X} = \hat{V}^{x^k} \frac{\partial}{\partial x^k} + D_\alpha(\hat{V}^{x^k}) \frac{\partial}{\partial x_\alpha^k} + D_\alpha D_\beta(\hat{V}^{x^k}) \frac{\partial}{\partial x_{\alpha\beta}^k} + \cdots, \tag{3.12}$$

where $D_0 = \partial/\partial t$ $D_i = \partial/\partial x_0^i$ denote total partial derivatives with respect to t and x_0^i $(1 \leqslant i \leqslant 3)$, $V^0 \equiv V^t$ and $V^i \equiv V^{x_0^i}$ respectively. $\hat{X}$ is the extended Lie symmetry operator corresponding to the canonical Lie transformation $x'^k = x^k + \epsilon\hat{V}^{x^k}$, $t' = t$ and $x_0'^j = x_0^j$.

Comment 2

The basic conservation law (3.4) and condition (3.9) for the action to be invariant under a divergence symmetry are a consequence of the identity

$$\tilde{X}\mathcal{L}^0 + \mathcal{L}^0 D_\alpha V^\alpha + D_\alpha \Lambda_0^\alpha = \hat{V}^{x^i} E_{x^i}(\mathcal{L}^0) + D_\alpha(W^\alpha + \mathcal{L}^0 V^\alpha) + D_\alpha \Lambda_0^\alpha, \tag{3.13}$$

where $E_{x^i}(\mathcal{L}^0) = \delta A/\delta x^i$ is the variational derivative of A with respect to x^i in (2.18) and $W^\alpha = \hat{V}^{x^k}\partial\mathcal{L}^0/\partial x_\alpha^k$ is a surface vector term that arises in the proof of Noether's theorem. Identity (3.13) is discussed in further detail in appendix B.

To convert the Lagrangian conservation law (3.4) to its equivalent Eulerian form we use a result of Padhye (1998) given below.

Theorem 3.1. *The Lagrangian conservation law (3.4) can be written as an Eulerian conservation law of the form*

$$\frac{\partial F^0}{\partial t} + \frac{\partial F^j}{\partial x^j} = 0, \tag{3.14}$$

where

$$F^0 = \frac{I^0}{J}, \qquad F^j = \frac{u^j I^0 + x_{jk} I^k}{J}, \quad (j = 1, 2, 3), \tag{3.15}$$

are the conserved density F^0 and flux components F^j.

Proposition 3.2. *The Lagrangian conservation law (3.4) with conserved density I^0 of (3.5), and flux I^j of (3.6), is equivalent to the Eulerian conservation law:*

$$\frac{\partial F^0}{\partial t} + \frac{\partial F^j}{\partial x^j} = 0, \tag{3.16}$$

where

$$F^0 = \rho u^k \hat{V}^{x^k(\mathbf{x}_0,t)} + V^t\mathcal{L} + \Lambda^0, \tag{3.17}$$

$$F^j = \hat{V}^{x^k(\mathbf{x}_0,t)}(T^{jk} - \mathcal{L}\delta^{jk}) + V^{x^j}\mathcal{L} + \Lambda^j, \tag{3.18}$$

$$T^{jk} = \rho u^j u^k + \left(p + \frac{B^2}{2\mu}\right)\delta^{jk} - \frac{B^j B^k}{\mu}, \tag{3.19}$$

$$\Lambda^0 = \frac{\Lambda_0^0}{J}, \qquad \Lambda^j = \frac{u^j \Lambda_0^0 + x_{js}\Lambda_0^s}{J}. \tag{3.20}$$

In (3.16)–(3.20) T^{jk} is the Eulerian momentum flux tensor (i.e. the spatial components of the stress energy tensor) and $\hat{V}^{x^k(\mathbf{x}_0,t)}$ is the canonical symmetry generator (3.6).

Comment

Padhye and Morrison (1996a, 1996b) and Padhye (1998) used theorem 3.1 to convert Lagrangian conservation laws to Eulerian conservation laws. Webb *et al* (2005b) derived Lagrangian and Eulerian conservation laws associated with a given Lie symmetry using propositions 3.1 and 3.2, including the effects of fully nonlinear MHD waves in a non-uniform and time-dependent background flow, based on the MHD action principle. Linear waves in a non-uniform background flow were studied in Webb *et al* (2005a), thus extending similar work by Dewar (1970) for WKB waves.

3.1. Fluid relabelling symmetries

Consider infinitesimal Lie transformations of the form (3.2) and (3.3), with

$$V^t = 0, \qquad V^{\mathbf{x}} = 0, \qquad V^{\mathbf{x}_0} \neq 0, \qquad \Lambda_0^\alpha = 0, \tag{3.21}$$

which leave action (3.1) invariant. The extended Lie transformation operator $\tilde{X}$ for the case (3.21) has generators:

$$\begin{aligned} &\hat{V}^{\mathbf{x}} = -V^{\mathbf{x}_0} \cdot \nabla_0 \mathbf{x}, \qquad V^{\mathbf{x}_t} = -D_t(V^{\mathbf{x}_0}) \cdot \nabla_0 \mathbf{x}, \\ &V^{\nabla_0 \mathbf{x}} = -\nabla_0(V^{\mathbf{x}_0}) \cdot \nabla_0 \mathbf{x}. \end{aligned} \tag{3.22}$$

Condition (3.9) for a divergence symmetry of the action reduces to

$$\begin{aligned} &\nabla_0 \cdot (\rho_0 V^{\mathbf{x}_0}) \left(\frac{1}{2}|\mathbf{u}|^2 - \Phi(\mathbf{x}) - \frac{\varepsilon + p}{\rho} \right) - J \frac{\partial \varepsilon(\rho, S)}{\partial S} V^{\mathbf{x}_0} \cdot \nabla_0 S \\ &\quad - D_t(\rho_0 V^{\mathbf{x}_0}) \cdot \nabla_0 \mathbf{x} \cdot \mathbf{u} - \frac{1}{\mu J} (\nabla_0 \mathbf{x}) \cdot (\nabla_0 \mathbf{x})^T : [(V^{\mathbf{x}_0} \cdot \nabla_0 \mathbf{B}_0) \mathbf{B}_0 \\ &\quad + \mathbf{B}_0 \mathbf{B}_0 \nabla_0 \cdot V^{\mathbf{x}_0} - (\mathbf{B}_0 \cdot \nabla_0 V^{\mathbf{x}_0}) \mathbf{B}_0] = 0. \end{aligned} \tag{3.23}$$

Equations (3.23) are satisfied if $V^{\mathbf{x}_0}$, $\mathbf{B}_0$, ρ_0 and S satisfy the equations

$$\begin{aligned} &\nabla_0 \cdot (\rho_0 V^{\mathbf{x}_0}) = 0, \qquad V^{\mathbf{x}_0} \cdot \nabla_0 S = 0, \qquad D_t(\rho_0 V^{\mathbf{x}_0}) = 0, \\ &\nabla_0 \times (V^{\mathbf{x}_0} \times \mathbf{B}_0) = 0, \qquad \nabla_0 \cdot \mathbf{B}_0 = 0. \end{aligned} \tag{3.24}$$

Equations (3.24) are the Lie determining equations for the fluid relabelling symmetries obtained by Padhye (1998) and Webb *et al* (2005b). These equations apply for a general equation of state for the gas with $\varepsilon = \varepsilon(\rho, S)$ and also apply in an external gravitational field described by the gravitational potential Φ. The fluid relabelling symmetries do not change the Eulerian physical variables ρ, $\mathbf{u}$, S, p, $\mathbf{B}$ under the Lie transformation (3.21).

3.2. The Galilei group

In this section we derive the Lagrangian symmetry operators that correspond to the Galilei group.

Proposition 3.3. *The Galilei group Eulerian symmetries P_μ, K_i and J_k in (1.12) correspond to the Lagrangian symmetry operators:*

$$\begin{aligned} &P_\mu^L = \frac{\partial}{\partial x^\mu} + V^{\mathbf{x}_0} \cdot \nabla_0, \qquad K_i^L = t \frac{\partial}{\partial x^i} + V^{\mathbf{x}_0} \cdot \nabla_0, \\ &J_k^L = \epsilon_{kij} x^j \frac{\partial}{\partial x^i} + V^{\mathbf{x}_0} \cdot \nabla_0, \end{aligned} \tag{3.25}$$

where $V^{\mathbf{x}_0}$ satisfies the fluid relabelling symmetry equations (3.24).

Proof. We illustrate the method of proof for the time translation symmetry operator $P_0 = \partial/\partial t$. Consider the time translation symmetry $X_1 = \partial/\partial t$ of (1.1) with Eulerian symmetry generators

$$V^t = 1, \qquad V^{x^i} = 0, \qquad V^{u^i} = V^{B^i} = V^p = V^\rho = 0. \tag{3.26}$$

Using the Lie extension formula

$$V^{x_t^i} = D_t\left(V^{x^i}\right) - D_t(V^t) x_t^i - D_t(V^{x_0^s}) x_{is}, \tag{3.27}$$

for $V^{u^i} \equiv V^{x_t^i}$ in Lagrange label space, we find

$$V^{u^i} = -D_t\left(V^{x_0^s}\right) x_{is} = 0 \qquad \text{or} \qquad D_t\left(V^{x_0^s}\right) = 0. \tag{3.28}$$

Similarly, the Lie extension formula

$$V^{x_{ij}} = D_{x_0^j}(V^{x^i}) - D_{x_0^j}(V^t)x_t^i - D_{x_0^j}(V^{x_0^s})x_{is}, \tag{3.29}$$

and (3.26) imply

$$V^{x_{ij}} = -D_{x_0^j}\left(V^{x_0^s}\right)x_{is}. \tag{3.30}$$

From the equation $V^\rho = 0$ we deduce

$$V^\rho = \tilde{X}(\rho) \equiv \tilde{X}\left(\frac{\rho_0}{J}\right) = \frac{1}{J}\nabla_0\cdot(\rho_0 V^{\mathbf{x}_0}) = 0. \tag{3.31}$$

Because $\bar{S} = \bar{S}(\mathbf{x}_0) = f(p,\rho)$ is the general equation of state for the gas, we find

$$\tilde{X}_1\bar{S} = f_p V^p + f_\rho V^\rho \equiv V^{\mathbf{x}_0}\cdot\nabla_0\bar{S} = 0. \tag{3.32}$$

Noting that $B^i = x_{ij}B_0^j/J$ from (2.9) the condition $V^{B^i} = 0$ in (3.26) implies

$$\begin{aligned} V^{B^i} &= \frac{x_{ij}}{J}[V^{\mathbf{x}_0}\cdot\nabla_0\mathbf{B}_0 + \mathbf{B}_0\nabla_0\cdot V^{\mathbf{x}_0} - \mathbf{B}_0\cdot\nabla_0 V^{\mathbf{x}_0}]^j \\ &\equiv -\frac{x_{ij}}{J}[\nabla_0\times(V^{\mathbf{x}_0}\times\mathbf{B}_0) - V^{\mathbf{x}_0}\nabla_0\cdot\mathbf{B}_0]^j = 0. \end{aligned} \tag{3.33}$$

Noting that we require $\nabla_0\cdot\mathbf{B}_0 = 0$ to ensure that $\nabla\cdot\mathbf{B} = 0$, (3.28), (3.31)–(3.33), require that $V^{\mathbf{x}_0}$ and $\mathbf{B}_0$ must satisfy the Lie determining equations (3.24) for the relabelling symmetries. The method of proof is similar for the other Eulerian Lie point symmetries in (1.12) or (1.1)–(1.5). □

3.3. The projective symmetry X_{14} of gas dynamics

Proposition 3.4. *The Eulerian, projective symmetry operator X_{14} of gas dynamics, for a gas with an adiabatic index of $\gamma = (n+2)/n$ (n is the number of Cartesian space dimensions), has the Lagrangian symmetry operator*

$$X_{14}^L = tx^\mu\frac{\partial}{\partial x^\mu} + V^{\mathbf{x}_0}\cdot\nabla_0, \tag{3.34}$$

where $V^{\mathbf{x}_0}$ satisfies the fluid relabelling symmetry determining equations with $\mathbf{B}_0 = 0$.

Proof. Consider the projective symmetry X_{14} in (1.11) for ideal, gas dynamics with $\gamma = (n+2)/n$. The Eulerian version of this symmetry has Lie symmetry generators:

$$\begin{aligned} &V^t = t^2, \qquad V^{x^i} = tx^i, \qquad V^{u^i} = x^i - tu^i, \\ &V^\rho = -nt\rho, \qquad V^p = -(n+2)tp. \end{aligned} \tag{3.35}$$

Following the procedure to convert symmetries from Eulerian to the Lagrangian form used in proposition 3.3, we find that $V^{\mathbf{x}_0}$ must satisfy the fluid relabelling determining equations (3.24) with $\mathbf{B}_0 = 0$. We derive the conservation law associated with X_{14} in section 6 (see also Ibragimov (1994)). The projective symmetry in 2D MHD, with an ignorable Cartesian space coordinate z, with $\mathbf{B} = (0, 0, B)$ is described in appendix C (see also Fuchs and Richter (1987)). □

4. Lagrangian symmetries for $X_{(a)}$

There are two tasks involved before one can establish conservation laws associated with the symmetry operator $X_{(a)}$ in (1.13). First, it is necessary to convert the symmetry into its Lagrangian form, and second it is necessary to determine the conditions for which the symmetry is a divergence symmetry or a variational symmetry of the MHD action, by investigating the conditions under which (3.9) is satisfied. After these two tasks are accomplished, the conservation laws may be determined by the use of Noether's first theorem.

In the following analysis we neglect the effect of gravity and set $\Phi = 0$.

4.1. The symmetry operator $X_{(a)}$

The symmetry operator $X_{(a)}$ from (1.13) and (1.8)–(1.10) can be written in the form

$$X_{(a)} = V^t \frac{\partial}{\partial t} + V^{x^i} \frac{\partial}{\partial x^i} + V^\rho \frac{\partial}{\partial \rho} + V^{u^i} \frac{\partial}{\partial u^i} + V^p \frac{\partial}{\partial p} + V^{B^i} \frac{\partial}{\partial B^i}, \tag{4.1}$$

where

$$\begin{aligned} &V^t = \alpha t, \qquad V^{x^i} = \beta x^i, \qquad V^{u^i} = (\beta - \alpha) u^i, \qquad V^{B^i} = \delta B^i, \\ &V^\rho = (2\alpha + 2\delta - 2\beta)\rho, \qquad V^p = 2\delta p. \end{aligned} \tag{4.2}$$

To reduce the symmetry operator $X_{(a)}$ to the Lagrangian form entails writing the extended Lie transformation operator in the form

$$\tilde{X}_{(a)} = V^t \frac{\partial}{\partial t} + V^{x^i} \frac{\partial}{\partial x^i} + V^{x_0^s} \frac{\partial}{\partial x_0^s} + V^{x_t^i} \frac{\partial}{\partial x_t^i} + V^{x_{ij}} \frac{\partial}{\partial x_{ij}} + \cdots, \tag{4.3}$$

where $x^i = x^i(\mathbf{x}_0, t)$ is regarded as a function of $\mathbf{x}_0$ and t.

Proposition 4.1. *The Lagrangian, Lie symmetry generator $V^{\mathbf{x}_0}$ in (4.3) corresponding to the Eulerian Lie operator of (4.1) and (4.2) satisfies the Lie determining equations:*

$$\nabla_0 \cdot (\rho_0 V^{\mathbf{x}_0}) = \rho_0[2\alpha + 2\delta + (n-2)\beta], \qquad D_t(V^{\mathbf{x}_0}) = 0, \tag{4.4}$$

$$V^{\bar{S}} = V^{\mathbf{x}_0} \cdot \nabla_0 \bar{S} = 2[\delta(1-\gamma) + \gamma(\beta - \alpha)], \tag{4.5}$$

$$\nabla_0 \times (V^{\mathbf{x}_0} \times \mathbf{B}_0) = -[\delta + \beta(n-1)]\mathbf{B}_0, \tag{4.6}$$

$$\nabla_0 \cdot \mathbf{B}_0 = 0, \tag{4.7}$$

where $\bar{S} = S/C_v$. The entropy constraint equation (4.5) is equivalent to

$$V^{\mathbf{x}_0} \cdot \nabla_0 p_0 + \gamma p_0 \nabla_0 \cdot V^{\mathbf{x}_0} = p_0(2\delta + n\gamma\beta). \tag{4.8}$$

Equation (4.8) is a result of a combination of the first equation (4.4) and the entropy generator equation (4.5).

Proof. The proof is based on the Lie extension formulae for the transformation of derivatives of $\mathbf{x}(\mathbf{x}_0, t)$. Using the Lie extension formula

$$V^{x_t^i} = D_t(V^{x^i}) - D_t(V^t) x_t^i - D_t(V^{x_0^s}) x_{is}, \tag{4.9}$$

we find

$$V^{u^i} = (\beta - \alpha) u^i \equiv V^{x_t^i} = (\beta - \alpha) x_t^i - D_t\left(V^{x_0^s}\right) x_{is}, \tag{4.10}$$

and hence

$$D_t\left(V^{x_0^s}\right) = 0, \tag{4.11}$$

is a constraint on $V^{x_0^s}$ for a consistent solution. Similarly, using the extension formula

$$V^{x_{ij}} = D_{x_0^j}\left(V^{x^i}\right) - D_{x_0^j}(V^t)x_t^i - D_{x_0^j}\left(V^{x_0^s}\right)x_{is}, \tag{4.12}$$

in conjunction with the expressions for V^{x^i} and V^t from (4.2) gives

$$V^{x_{ij}} = \beta x_{ij} - D_{x_0^j}\left(V^{x_0^s}\right)x_{is}, \tag{4.13}$$

for $V^{x_{ij}}$.

Next using the Lagrangian continuity equation (2.7) and the symmetry generator V^ρ in (4.2) we obtain

$$V^\rho = \tilde{X}_{(a)}\rho \equiv \tilde{X}_{(a)}\left(\frac{\rho_0}{J}\right) = (2\alpha + 2\delta - 2\beta)\frac{\rho_0}{J}. \tag{4.14}$$

However, using (4.13) gives

$$\tilde{X}_{(a)}J = V^{x_{ij}}\frac{\partial J}{\partial x_{ij}} = V^{x_{ij}}A_{ij} = (n\beta - \nabla_0 \cdot V^{\mathbf{x}_0})J, \tag{4.15}$$

where n is the number of independent $\mathbf{x}_0$ labels. Using (4.15) in (4.14) gives

$$\nabla_0 \cdot \left(\rho_0 V^{\mathbf{x}_0}\right) = \rho_0\left[2\alpha + 2\delta + (n-2)\beta\right], \tag{4.16}$$

where $\nabla_0 \equiv \partial/\partial\mathbf{x}_0$ denotes the gradient operator in $\mathbf{x}_0$ label space. In the derivation of (4.15) use has been made of Cramer's rule $x_{ij}A_{is} = J\delta_{js}$. Equations (4.16) and (4.11) are the two equations listed in (4.4). These equations are reminiscent of the continuity equation for steady flow, with a source, in which $V^{\mathbf{x}_0}$ plays the role of the fluid velocity in $\mathbf{x}_0$ label space.

Next consider the constraint that the entropy $S = S(\mathbf{x}_0)$. Using $\bar{S} = S/C_v = \ln(p/\rho^\gamma)$, and V^p and V^ρ from (4.12) gives

$$V^{\bar{S}} = \tilde{X}_{(a)}\bar{S} = V^{\mathbf{x}_0} \cdot \nabla_0\bar{S} = \frac{V^p}{p} - \gamma\frac{V^\rho}{\rho} = 2\left[\delta(1-\gamma) + \gamma(\beta - \alpha)\right]. \tag{4.17}$$

This is (4.5) for $V^{\bar{S}}$. Using $\bar{S} = \ln(p_0/\rho_0^\gamma)$ is solely a function of the Lagrange label $\mathbf{x}_0$, (4.17) coupled with the 'continuity' equation (4.16) for $V^{\mathbf{x}_0}$ implies

$$V^{\mathbf{x}_0} \cdot \nabla_0 p_0 + \gamma p_0 \nabla_0 \cdot V^{\mathbf{x}_0} = p_0(2\delta + n\gamma\beta), \tag{4.18}$$

as the alternative form (4.8) of the constraint imposed by requiring that $S = S(\mathbf{x}_0)$.

Consider the Lagrangian form of the Lie transformation for the magnetic field. From the frozen in field theorems (2.9) and (2.13) we obtain $B^i = x_{ik}B_0^k/J \equiv B_{ki}B_0^k$. Because $\partial B_{ki}/\partial x^i = 0$ we obtain

$$\nabla \cdot \mathbf{B} = \frac{\partial B^i}{\partial x^i} = B_{ki}\frac{\partial B_0^k}{\partial x^i} = \frac{1}{J}\nabla_0 \cdot \mathbf{B}_0. \tag{4.19}$$

Thus, to ensure $\nabla \cdot \mathbf{B} = 0$ we require $\nabla_0 \cdot \mathbf{B}_0 = 0$ as in (4.7).

From the Eulerian Lie transformation equation $V^{B^i} = \delta B^i$ in (4.2) we have

$$V^{B^i} = \tilde{X}_{(a)}B^i = \tilde{X}_{(a)}\left(\frac{x_{ij}B_0^j}{J}\right) = \delta B^i. \tag{4.20}$$

Using the Lagrangian form of the Lie transformation operator $\tilde{X}$ from (4.3), and result (4.13) for $V^{x_{ij}}$, (4.20) reduces to

$$\frac{x_{ij}}{J}\{V^{\mathbf{x}_0} \cdot \nabla_0\mathbf{B}_0 - \mathbf{B}_0 \cdot \nabla_0 V^{\mathbf{x}_0} + \nabla_0 \cdot V^{\mathbf{x}_0} + [(1-n)\beta - \delta]\mathbf{B}_0\}^j = 0. \tag{4.21}$$

Noting that $\nabla_0 \cdot \mathbf{B}_0 = 0$ and assuming $J \neq 0$, (4.21) may be re-written in the form (4.6). This completes the proof. □

Proposition 4.2. *Condition (3.9) for $\tilde{X}_{(a)}$ to be a variational symmetry of the MHD action (3.1), in which the symmetry satisfies the Lie determining equations (4.4)–(4.8) may be written in the form*

$$\tilde{X}_{(a)}\mathcal{L}^0 + (D_t V^t + \nabla_0 \cdot V^{\mathbf{x}_0})\mathcal{L}^0 = (\alpha + 2\delta + n\beta)\mathcal{L}^0. \tag{4.22}$$

Thus, $\tilde{X}_{(a)}$ is a variational symmetry of action (3.1) if

$$\alpha + 2\delta + n\beta = 0, \tag{4.23}$$

and $V^{\mathbf{x}_0}$ satisfies (4.4)–(4.8). In the latter case, Noether's theorem (propositions 3.1 and 3.2) guarantees the existence of a Lagrangian conservation law of the form (3.4) and an equivalent Eulerian conservation law of the form (3.16) associated with the symmetry.

Proof. Evaluation of the left-hand side of (4.22), where $\mathcal{L}^0$ is given by (2.17) and using the Lie determining equations (4.4)–(4.8), gives result (4.22). Thus, if condition (4.23) is satisfied, the symmetry is a variational symmetry of the action. □

4.2. Determining equation solutions

In this section we obtain solutions of the Lie determining equations (4.4)–(4.8).

Proposition 4.3. *The Lie determining equations (4.4) and (4.6) for $V^{\mathbf{x}_0}$ and $\mathbf{B}_0$ can be re-written in the form*

$$\mathbf{b}_0 \cdot \nabla_0 V^{\mathbf{x}_0} - V^{\mathbf{x}_0} \cdot \nabla_0 \mathbf{b}_0 = \delta_4 \mathbf{b}_0, \tag{4.24}$$

where

$$\mathbf{b}_0 = \frac{\mathbf{B}_0}{\rho_0} \qquad \text{and} \qquad \delta_4 = \delta + 2\alpha - \beta. \tag{4.25}$$

Equation (4.24) for $\delta_4 \neq 0$ defines a two-dimensional, non-Abelian Lie algebra for the vector fields

$$X_1 = b_0^j \frac{\partial}{\partial x_0^j}, \qquad X_2 = V^{x_0^j} \frac{\partial}{\partial x_0^j}, \tag{4.26}$$

with Lie bracket $[X_\alpha, X_\beta]$ satisfying the commutation relation

$$[X_1, X_2] = \delta_4 X_1. \tag{4.27}$$

From Frobenius's theorem, the set of vector fields $\{X_\alpha : \alpha = 1, 2\}$ form a Lie algebra, and hence an integrable manifold $\mathbf{x}_0 = \mathbf{x}_0(y_1, y_2)$, and the vector fields are said to be in involution. The two-dimensional Lie algebra $\{X_1, X_2\}$ defined by (4.27) has the representation

$$X_1 = \delta_4 y_1 \frac{\partial}{\partial y_2}, \qquad X_2 = -\delta_4 y_1 \frac{\partial}{\partial y_1}, \tag{4.28}$$

where $y_1(\mathbf{x}_0)$ and $y_2(\mathbf{x}_0)$ are independent functions of $\mathbf{x}_0$. The vector fields $\mathbf{b}_0$ and $V^{\mathbf{x}_0}$ have solutions

$$\mathbf{b}_0 = \delta_4 y_1 \mathbf{e}_2, \qquad V^{\mathbf{x}_0} = -\delta_4 y_1 \mathbf{e}_1, \qquad \mathbf{e}_1 = \frac{\partial \mathbf{x}_0}{\partial y_1}, \qquad \mathbf{e}_2 = \frac{\partial \mathbf{x}_0}{\partial y_2}, \tag{4.29}$$

where $y_1(\mathbf{x}_0)$ and $y_2(\mathbf{x}_0)$ are independent functions of $\mathbf{x}_0$, and $\mathbf{e}_1$ and $\mathbf{e}_2$ are holonomic base vectors normal to the $y_1 = \text{const}$ and $y_2 = \text{const}$ surfaces.

Proof. From (4.6)–(4.7),

$$\mathbf{B}_0 \cdot \nabla_0 V^{\mathbf{x}_0} - \mathbf{B}_0 \nabla_0 \cdot V^{\mathbf{x}_0} - V^{\mathbf{x}_0} \cdot \nabla_0 \mathbf{B}_0 = -[\delta + \beta(n-1)]\mathbf{B}_0. \tag{4.30}$$

Solving the 'continuity' equation (4.4) for $\nabla \cdot V^{\mathbf{x}_0}$ gives

$$\nabla_0 \cdot V^{\mathbf{x}_0} = -V^{\mathbf{x}_0} \cdot \nabla_0 \ln \rho_0 + [2\alpha + 2\delta + (n-2)\beta]. \tag{4.31}$$

Using (4.31) for $\nabla_0 \cdot V^{\mathbf{x}_0}$ in (4.30) gives result (4.24) relating $V^{\mathbf{x}_0}$ and $\mathbf{b}_0 = \mathbf{B}_0/\rho_0$. Next we note that the Lie bracket of the vector fields X_1 and X_2 in (4.26) is defined as

$$[X_1, X_2] = X_1 X_2 - X_2 X_1 = \left(\mathbf{b}_0 \cdot \nabla_0 V^{x_0^i} - V^{\mathbf{x}_0} \cdot \nabla_0 b_0^i\right) \frac{\partial}{\partial x_0^i}, \tag{4.32}$$

from which it follows that (4.24) can be written as the Lie bracket equation (4.27).

Note that the vector fields $\{X_\alpha : \alpha = 1, 2\}$ in (4.28) have the form $X_\alpha = C^\mu_{\alpha\nu} y_\mu \partial/\partial y_\nu$, where $C^\mu_{\alpha\nu}$ are the structure constants of the Lie algebra. From (4.29) and (4.26) it follows that

$$b_0^s = \delta_4 y_1 \frac{\partial x_0^s}{\partial y_2}, \qquad V^{x_0^s} = -\delta_4 y_1 \frac{\partial x_0^s}{\partial y_1}, \tag{4.33}$$

are solutions for b_0^s and $V^{x_0^s}$ satisfying (4.24). These solutions for $\mathbf{b}_0$ and $V^{\mathbf{x}_0}$ can be written in the more geometrically revealing form (4.29). This completes the proof. □

Comment

The Frobenius integrability theorem guarantees that for given vector fields $\mathbf{b}_0$ and $V^{\mathbf{x}_0}$, the differential equations (4.33) for $\mathbf{x}_0(y_1, y_2)$ are integrable, as a consequence of the fact that the vector fields X_1 and X_2 form a two-dimensional Lie algebra defined by (4.27). This is easily demonstrated from differentiating (4.33) with respect to y_1 and y_2 and requiring that the solution has continuous second-order derivatives.

Proposition 4.4

$$\begin{aligned} &\mathbf{b}_0 \cdot \nabla_0 y_1 = 0, && \mathbf{b}_0 \cdot \nabla_0 y_2 = \delta_4 y_1, \\ &V^{\mathbf{x}_0} \cdot \nabla_0 y_1 = -\delta_4 y_1, && V^{\mathbf{x}_0} \cdot \nabla_0 y_2 = 0. \end{aligned} \tag{4.34}$$

Proof. Results (4.34) follow by noting $X_1 = \mathbf{b}_0 \cdot \nabla_0$ and $X_2 = V^{\mathbf{x}_0} \cdot \nabla_0$ and by using representation (4.28) for X_1 and X_2. Note that (4.34) implies that $\mathbf{B}_0$ lies in the $y_1 = \text{const}$ surface, and $V^{\mathbf{x}_0}$ lies in the $y_2 = \text{const}$ surface. □

4.2.1. Geometrical aspects. So far, the analysis of (4.4) and (4.6) has revealed that $V^{\mathbf{x}_0}$ and $\mathbf{b}_0$ lie in the surface $\mathbf{x}_0 = \mathbf{x}_0(y_1, y_2, c)$, where $c = \text{const}$. The family of surfaces $\mathbf{x}_0 = \mathbf{x}_0(y_1, y_2, c)$ for different c can also be written as $y_3 = \text{const}$, where $y_3(\mathbf{x}_0)$ is independent of y_1 and y_2. Thus, we can use

$$\mathbf{e}_1 = \frac{\partial \mathbf{x}_0}{\partial y_1}, \qquad \mathbf{e}_2 = \frac{\partial \mathbf{x}_0}{\partial y_2}, \qquad \mathbf{e}_3 = \frac{\partial \mathbf{x}_0}{\partial y_3}, \tag{4.35}$$

as holonomic base vectors. As $\mathbf{e}_1 \cdot \nabla_0 y_3 = \partial y_3/\partial y_1 = 0$ and $\mathbf{e}_2 \cdot \nabla_0 y_3 = \partial y_3/\partial y_2 = 0$ then

$$\mathbf{e}_1 \times \mathbf{e}_2 = \lambda \mathbf{e}_3, \qquad \lambda = \frac{\sqrt{g}}{g_{33}}. \tag{4.36}$$

In the above development, $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ is the metric tensor, associated with the generalized coordinates (y_1, y_2, y_3). It has the form

$$\mathbf{g} = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & g_{33} \end{pmatrix}, \tag{4.37}$$

and $g = \det(g_{\alpha\beta})$ is the determinant of the metric tensor. The inverse matrix, $\mathbf{g}^{-1}$, has components $g^{\alpha\beta}$ given by the matrix

$$\mathbf{g}^{-1} = \frac{1}{g}\begin{pmatrix} g_{22}g_{33} & -g_{12}g_{33} & 0 \\ -g_{12}g_{33} & g_{11}g_{33} & 0 \\ 0 & 0 & (g_{11}g_{22} - g_{12}g_{21}) \end{pmatrix}. \tag{4.38}$$

Note that $g^{\alpha\beta} = G_{\beta\alpha}/g$, where $G_{\alpha\beta} = \mathrm{cofac}(g_{\alpha\beta})$. In the derivation of (4.36) we note that

$$\mathbf{e}_1 \times \mathbf{e}_2 \cdot \mathbf{e}_3 = \lambda g_{33} = \det\left(\partial x_0^\alpha / \partial y_j\right) = \sqrt{g}, \tag{4.39}$$

and hence $\lambda = \sqrt{g}/g_{33}$.

Using the dual base vectors $\omega^i = \nabla_0 y^i$, $(1 \leqslant i \leqslant 3)$, we obtain

$$\begin{aligned} &\omega^1 \times \omega^2 = \mathbf{e}_3/\sqrt{g}, \qquad && \omega^2 \times \omega^3 = \mathbf{e}_1/\sqrt{g}, \qquad && \omega^3 \times \omega^1 = \mathbf{e}_2/\sqrt{g}, \\ &\mathbf{e}_1 \times \mathbf{e}_2 = \sqrt{g}\omega^3, && \mathbf{e}_2 \times \mathbf{e}_3 = \sqrt{g}\omega^1, && \mathbf{e}_3 \times \mathbf{e}_1 = \sqrt{g}\omega^2. \end{aligned} \tag{4.40}$$

These relations allow one to express solutions of the Lie determining equations (4.4)–(4.8) in either the contravariant vector field base $\{\mathbf{e}_i\}$ or in terms of the covariant vector field basis $\{\omega^i\}$.

4.2.2. Solutions of (4.4)–(4.7).

Proposition 4.5. *The Lie determining equations (4.4)–(4.7) have solutions for* $\mathbf{B}_0$, $V^{\mathbf{x}_0}$, ρ_0 *and* $\bar{S}(\mathbf{x}_0)$ *that depend on the parameter* $\delta_4 = \delta + 2\alpha - \beta$.

Case 1. $\delta_4 \neq 0$

If $\delta_4 \neq 0$, (4.4)–(4.7) have solutions of the form

$$\begin{aligned} &\mathbf{B}_0 = \delta_4 \rho_0 y_1 \mathbf{e}_2, \qquad && V^{\mathbf{x}_0} = -\delta_4 y_1 \mathbf{e}_1, \\ &\rho_0 = \frac{F(y_3)}{\sqrt{g}} |y_1|^{-(1+\delta_2/\delta_4)}, && \bar{S} = -2\left(\frac{\delta_1}{\delta_4}\right) \ln|y_1| + D(y_2, y_3), \end{aligned} \tag{4.41}$$

where

$$\delta_1 = \delta(1-\gamma) + \gamma(\beta - \alpha), \qquad \delta_2 = 2\alpha + 2\delta + (n-2)\beta, \qquad \delta_4 = \delta + 2\alpha - \beta. \tag{4.42}$$

In (4.41), (y_1, y_2, y_3) are the coordinates in $\mathbf{x}_0$ label space and $\mathbf{e}_j = \partial \mathbf{x}_0/\partial y_j$ are the corresponding holonomic base vectors referred to in (4.35) *et seq*. $F(y_3)$ and $D(y_2, y_3)$ are arbitrary functions of y_2 and (y_2, y_3) respectively that arise as 'integration constants'. Here $g = \det(g_{ij})$ is the determinant of the metric tensor $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$.

Case 2. $\delta_4 = 0$

If $\delta_4 = 0$, (4.4)–(4.7) have solutions

$$\begin{aligned} &\mathbf{B}_0 = \rho_0 \mathbf{e}_2, \qquad && V^{\mathbf{x}_0} = \mathbf{e}_1, \\ &\rho_0 = \frac{F(y_3)}{\sqrt{g}} \exp(\delta_2 y_1), && \bar{S} = 2\delta_1 y_1 + D(y_2, y_3). \end{aligned} \tag{4.43}$$

The coordinates (y_1, y_2, y_3) are different than those used in the case $\delta_4 \neq 0$.

Proof. The proof of (4.41)–(4.43) is straightforward. One can check the solutions by substitution in (4.4)–(4.7). □

5. MHD conservation laws

In this section we use the results of section 4, to derive conservation laws for MHD associated with the variational symmetry (1.13) for the case, where $\alpha + 2\delta + n\beta = 0$. These conservation laws are derived using Noether's theorem, coupled with the solutions of the Lie determining equations (4.4)–(4.7) in Lagrange label space.

5.1. Conservation laws in $\mathbf{x}_0$ label space

Proposition 5.1. *If $\alpha + 2\delta + n\beta = 0$, the MHD system (2.1)–(2.6) with equation of state (1.7), in the absence of gravitational forces, admits the variational symmetry (1.13) which leaves the MHD action (2.15) invariant. The corresponding conservation law, by Noether's theorem (proposition 3.1) has the form*

$$\frac{\partial I^0}{\partial t} + \frac{\partial I^j}{\partial x_0^j} = 0, \tag{5.1}$$

where

$$I^0 = \rho_0 u^k \hat{V}^{x^k} + V^t \mathcal{L}^0, \tag{5.2}$$

$$I^j = \hat{V}^{x^k} \left[\left(p + \frac{B^2}{2\mu} \right) \delta^{ks} - \frac{B^k B^s}{\mu} \right] A_{sj} + V^{x_0^j} \mathcal{L}^0. \tag{5.3}$$

In (5.1)–(5.3)

$$\mathcal{L}^0 = \frac{1}{2}\rho_0 |\mathbf{u}|^2 - \frac{Jp}{\gamma - 1} - \frac{JB^2}{2\mu}, \tag{5.4}$$

is the Lagrangian density $\mathcal{L}^0$ in the MHD action principle (2.17) and $J = \det(\partial x^i/\partial x_0^j)$ is the Jacobian of the transformation between the Eulerian position of the fluid element $\mathbf{x}$ and the Lagrangian labels $\mathbf{x}_0$. The Lie symmetry generators $\hat{V}^{x^k}$ and $V^{x_0^j}$ depend on whether $\delta_4 = \delta + 2\alpha - \beta$ is non-zero or zero.

Case 1. $\delta_4 \neq 0$

In this case

$$\hat{V}^{x^k} = \beta x^k - \alpha t u^k + \delta_4 y_1 \frac{\partial x^k}{\partial y_1}, \qquad V^{x_0^j} = -\delta_4 y_1 \frac{\partial x_0^j}{\partial y_1}, \qquad V^t = \alpha t. \tag{5.5}$$

The associated solutions for $\mathbf{B}_0$, ρ_0 and $\bar{S}$ are given by (4.41) and (4.42).

Case 2. $\delta_4 = 0$

In this case

$$\hat{V}^{x^k} = \beta x^k - \alpha t u^k - \frac{\partial x^k(\mathbf{x}_0, t)}{\partial y_1}, \qquad V^{x_0^j} = \frac{\partial x_0^j}{\partial y_1}, \qquad V^t = \alpha t. \tag{5.6}$$

The associated solutions for $\mathbf{B}_0$, ρ_0 and $\bar{S}$ are given by (4.43).

Proof. The proof is a straightforward consequence of the Lagrangian form of Noether's theorem (proposition 3.1) and the solution of the Lie determining equations (4.4)–(4.7) in Lagrange label space given in proposition 4.5. □

Comment

The conservation law (5.1) depends on the initial distributions for ρ_0, $\mathbf{B}_0$, $\bar{S}$ in Lagrange label space as well as on the Lie symmetry generators in Lagrange label space. Below we show that one can use $\{y_j : 1 \leqslant j \leqslant 3\}$ to replace the Lagrange labels $\{x_0^j : 1 \leqslant j \leqslant 3\}$ to simplify the form of the conservation law (5.1).

5.2. Conservation laws in (y_1, y_2, y_3) label space

In order to transform the conservation law (5.1) into an equivalent conservation law in $\mathbf{y} = (y_1, y_2, y_3)^t$ Lagrange label space it is necessary to determine the nature of the transformation between $\mathbf{x}_0$ label space and $\mathbf{y}$ label space, as outlined below.

Proposition 5.2. *The determinant*

$$D = \det\left(\partial x_0^i / \partial y_j\right) = \sqrt{g}, \qquad \textit{where} \quad g = \det(g_{\alpha\beta}), \tag{5.7}$$

is the determinant of the metric tensor $g_{\alpha\beta} = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta$ *and the* $\mathbf{e}_\alpha = \partial \mathbf{x}_0 / \partial y_\alpha$ *are holonomic base vectors in Lagrange label space. Furthermore, the determinants*

$$J = \det\left(\partial x^i / \partial x_0^j\right) \qquad \text{and} \qquad \tilde{J} = \det\left(\partial x^i / \partial y_j\right), \tag{5.8}$$

are related by the equation

$$\tilde{J} = J\sqrt{g}. \tag{5.9}$$

Proof. The proof is straightforward and is omitted. □

Proposition 5.3. *The conservation law (5.1) in* $\mathbf{x}_0$ *label space can be written as the conservation law*

$$\frac{\partial I'^0}{\partial t} + \frac{\partial I'^j}{\partial y_j} = 0, \tag{5.10}$$

in $\mathbf{y}$ *label space, where*

$$I'^0 = \sqrt{g} I^0 \qquad \text{and} \qquad I'^j = \sqrt{g}\tilde{I}^j = \sqrt{g}\frac{\partial y_j}{\partial x_0^s} I^s. \tag{5.11}$$

Proof. The proof follows from the transformation of the vector field

$$\mathbf{I} = \sum_{k=1}^{3} I^k \mathbf{a}_k = \sum_{j=1}^{3} \tilde{I}^j \mathbf{e}_j, \tag{5.12}$$

where $\mathbf{a}_1 = (1, 0, 0)^T$, $\mathbf{a}_2 = (0, 1, 0)^T$ and $\mathbf{a}_3 = (0, 0, 1)^T$ are rectangular Cartesian base vectors in $\mathbf{x}_0$ label space and by noting that $\mathbf{e}_j = \partial \mathbf{x}_0 / \partial y_j$. □

Proposition 5.4. *The conservation law (5.10) of proposition 5.3 has conserved density* I'^0 *and flux components* I'^j *of the form*

$$I'^\alpha = I'^\alpha(t, \partial \mathbf{x}/\partial t, \mathbf{y}, \partial x^i / \partial y_j), \quad \alpha = 0, 1, 2, 3. \tag{5.13}$$

In other words, the flux components I'^α *can be written in terms of the generalized Lagrange labels* $\mathbf{y} = \mathbf{y}(\mathbf{x}_0)$*, without explicit reference to* $\mathbf{x}_0$*. The detailed form of the* I'^α *depends on whether* $\delta_4 = \delta + 2\alpha - \beta = 0$ *or* $\delta_4 \neq 0$*. Furthermore, it is necessary that* $\alpha + 2\delta + n\beta = 0$ *for the conservation law (5.10) to apply.*

Proof. Using propositions (5.1)–(5.3), we obtain

$$I'^0 = \tilde{\rho}_0 \hat{V}^{x^k} u^k + \alpha t \tilde{\mathcal{L}}^0, \tag{5.14}$$

$$I'^j = \hat{V}^{x^k} \left[\left(p + \frac{B^2}{2\mu} \right) \delta^{mk} - \frac{B^m B^k}{\mu} \right] \tilde{A}_{mj} + V^{y_j} \tilde{\mathcal{L}}^0, \tag{5.15}$$

where

$$\tilde{\mathcal{L}}^0 = \sqrt{g}\mathcal{L}^0 = \frac{1}{2}\tilde{\rho}_0 |\mathbf{x}_t|^2 - \frac{\tilde{p}_0 \tilde{J}^{1-\gamma}}{\gamma - 1} - \frac{\hat{B}_0^2}{2\mu \tilde{J}} \left| \frac{\partial \mathbf{x}}{\partial y_2} \right|^2, \tag{5.16}$$

$$\tilde{J} = \det(\partial x^i / \partial y_j), \qquad \tilde{A}_{mj} = \operatorname{cofac}(\partial x^m / \partial y_j). \tag{5.17}$$

The gas density ρ, pressure p and magnetic induction $\mathbf{B}$ are given by

$$\rho = \frac{\tilde{\rho}_0}{\tilde{J}}, \qquad p = \tilde{p}_0 \tilde{J}^{-\gamma}, \qquad \mathbf{B} = \frac{\hat{B}_0}{\tilde{J}} \frac{\partial \mathbf{x}}{\partial y_2}. \tag{5.18}$$

In (5.18)

$$\mathbf{B}_0 = B_0^j \mathbf{a}_j = \tilde{B}_0^k \mathbf{e}_k, \qquad \tilde{B}_0^k = \frac{\hat{B}_0 \delta^{k2}}{\sqrt{g}} = B_0^j \frac{\partial y_k}{\partial x_0^j}, \tag{5.19}$$

defines $\hat{B}_0$. The detailed form of $\tilde{\rho}_0$, $\tilde{p}_0$, $\hat{B}_0$, $\hat{V}^{x^k}$, $V^{x_0^j}$ and V^{y_j} depend on whether $\delta_4 = 0$ or $\delta_4 \neq 0$, which are given below.

Case 1. $\delta_4 \neq 0$

$$\begin{aligned}
&\hat{V}^{x^k} = \beta x^k - \alpha t u^k + \delta_4 y_1 \frac{\partial x^k}{\partial y_1}, \qquad V^{x_0^j} = -\delta_4 y_1 \frac{\partial x_0^j}{\partial y_1}, \qquad V^{y_j} = -\delta_4 y_1 \delta^{j1},\\
&\tilde{\rho}_0 = \rho_0 \sqrt{g} = F(y_3) |y_1|^{-(1+\delta_2/\delta_4)},\\
&\tilde{p}_0 = \exp(\bar{S}) \tilde{\rho}_0^{\gamma} = |y_1|^{-2\delta_1/\delta_4} \exp[D(y_2, y_3)] \tilde{\rho}_0^{\gamma},\\
&\hat{B}_0 = \delta_4 y_1 \tilde{\rho}_0.
\end{aligned} \tag{5.20}$$

Case 2. $\delta_4 = 0$

$$\begin{aligned}
&\hat{V}^{x^k} = \beta x^k - \alpha t u^k - \frac{\partial x^k}{\partial y_1}, \qquad V^{x_0^j} = \frac{\partial x_0^j}{\partial y_1}, \qquad V^{y_j} = \delta^{j1},\\
&\tilde{\rho}_0 = \rho_0 \sqrt{g} = F(y_3) \exp(\delta_2 y_1),\\
&\tilde{p}_0 = \exp(\bar{S}) \tilde{\rho}_0^{\gamma} = \exp[2\delta_1 y_1 + D(y_2, y_3)] \tilde{\rho}_0^{\gamma}, \qquad \hat{B}_0 = \tilde{\rho}_0.
\end{aligned} \tag{5.21}$$

In the derivation of (5.15) for I'^j we used the result

$$\sqrt{g} \frac{\partial y_j}{\partial x_0^s} A_{ms} = \sqrt{g} \frac{\partial y_j}{\partial x_0^s} \left(J \frac{\partial x_0^s}{\partial x^m} \right) = \sqrt{g} J \frac{\partial y_j}{\partial x^m} = \tilde{A}_{mj}, \tag{5.22}$$

(note $\tilde{J} = \sqrt{g} J$).

Inspection of (5.14)–(5.21) reveals that $\{I'^\alpha : \alpha = 0, 1, 2, 3\}$ depend only on $(t, \partial \mathbf{x}/\partial t, \mathbf{y}, \partial x^i / \partial y_j)$, and not explicitly on $\mathbf{x}_0$. This completes the proof. □

Remark 1. Note that V^{y_j}, $\tilde{\rho}_0$, $\hat{B}_0$, $\tilde{p}_0$ depend only on the Lagrange labels $\mathbf{y}$.

Remark 2. From (5.14) *et seq* one can show that

$$I'^0 = \hat{V}^{x^k} \frac{\partial \tilde{\mathcal{L}}^0}{\partial x_t^k} + V^t \tilde{\mathcal{L}}^0, \qquad I'^j = \hat{V}^{x^k} \frac{\partial \tilde{\mathcal{L}}^0}{\partial \tilde{x}_{kj}} + V^{y_j} \tilde{\mathcal{L}}^0, \tag{5.23}$$

where $\tilde{x}_{kj} = \partial x^k / \partial y_j$. One can verify that

$$\tilde{X}_{(a)} \tilde{\mathcal{L}}^0 + (D_t V^t + D_{y_j} V^{y_j}) \tilde{\mathcal{L}}^0 = (\alpha + 2\delta + n\beta) \tilde{\mathcal{L}}^0. \tag{5.24}$$

Equation (5.24) implies that the symmetry $X_{(a)}$ is a variational symmetry of the action if $\alpha + 2\delta + n\beta = 0$ (see also (4.22)). I'^0 and I'^j in (5.23) are the conserved density and fluxes that one obtains by the application of Noether's theorem if one uses (y_1, y_2, y_3) as the Lagrangian labels in the action principle.

6. Gas dynamics

In this section we discuss the solution of the Lie determining equations for the symmetry $X_{(a)}$ of (1.13) in Lagrange label space, and the form of the conservation laws that result if the condition $\alpha + 2\delta + n\beta = 0$ for a variational symmetry is satisfied.

In the pure fluid dynamical case, the Lie determining equations (4.4)–(4.7) reduce to the equation system

$$\nabla_0 \cdot (\rho_0 V^{\mathbf{x}_0}) = \delta_2 \rho_0, \qquad D_t (V^{\mathbf{x}_0}) = 0, \tag{6.1}$$

$$V^{\mathbf{x}_0} \cdot \nabla_0 \bar{S} = 2\delta_1, \tag{6.2}$$

where

$$\delta_1 = \delta(1 - \gamma) + \gamma(\beta - \alpha), \qquad \delta_2 = 2\alpha + 2\delta + (n - 2)\beta. \tag{6.3}$$

Proposition 6.1. *The Lie determining equations (6.1) and (6.2) for X for* $\delta_1 \neq 0$ *and* $\delta_2 \neq 0$ *have solutions*

$$V^{\mathbf{x}_0} = \mathbf{e}_1 = \frac{\partial \mathbf{x}_0}{\partial y_1}, \qquad \rho_0 = \frac{\Phi(y_2, y_3)}{\sqrt{g}} \exp(\delta_2 y_1), \qquad \bar{S} = 2\delta_1 y_1 + D(y_2, y_3), \tag{6.4}$$

where $g = \det(g_{ij})$, $\mathbf{e}_i = \partial \mathbf{x}_0 / \partial y_i$, y_1, y_2 *and* y_3 *are the three independent functions of the Lagrange labels* $\mathbf{x}_0$ *described in (4.35) et seq, and* $\Phi(y_2, y_3)$ *and* $D(y_2, y_3)$ *are arbitrary functions of* y_2 *and* y_3 *respectively.*

Proof. It is straightforward to verify that (6.4) provide a solution of (6.1) and (6.2). In (6.4) $V^{\mathbf{x}_0} = \partial \mathbf{x}_0 / \partial y_1$, with y_2 and y_3 fixed, can be regarded as the streamline equation for $V^{\mathbf{x}_0}$ in which y_1 is the parameter along the streamline. Solutions (6.4) for $V^{\mathbf{x}_0}$, ρ_0 and $\bar{S}$ are similar to solutions (4.43) of the MHD determining equations, except $\mathbf{B}_0 = 0$. □

6.1. Conservation laws

Proposition 6.2. *In the case* $\alpha + 2\delta + n\beta = 0$, *the Lagrangian conservation law (3.4) in* $(\mathbf{x}_0, t)$ *space has conserved density* I^0 *and fluxes* I^j *given by*

$$I^0 = \rho_0 u^k \hat{V}^{x^k} + \alpha t \mathcal{L}^0, \qquad I^j = p \hat{V}^{x^k} A_{kj} + V^{x_0^j} \mathcal{L}^0, \tag{6.5}$$

where $A_{kj} = \mathrm{cofac}(x_{kj})$, $x_{kj} = \partial x^k / \partial x_0^j$, *and*

$$\begin{aligned} \hat{V}^{x^k} &= \beta x^k - \alpha t u^k - \frac{\partial x^k}{\partial y_1}, \qquad & V^{x_0^j} &= \frac{\partial x_0^j}{\partial y_1}, \\ \mathcal{L}^0 &= \frac{1}{2} \rho_0 |\mathbf{u}|^2 - \frac{p_0 J^{1-\gamma}}{\gamma - 1}, \qquad & J &= \det(x_{ij}), \end{aligned} \tag{6.6}$$

define the symmetry generators $\hat{V}^{x^k}$, $V^{x_0^j}$, *Lagrangian* $\mathcal{L}^0$ *and the determinant* J *of the Euler–Lagrange map.*

The corresponding conservation law using $(\mathbf{y}, t)$ as independent variables from (5.14) and (5.15) has conserved density I'^0 and fluxes I'^j given by

$$I'^0 = \tilde{\rho}_0 \hat{V}^{x^k} u^k + \alpha t \tilde{\mathcal{L}}^0, \qquad I'^j = p \hat{V}^{x^k} \tilde{A}_{kj} + V^{y_j} \tilde{\mathcal{L}}^0, \tag{6.7}$$

where $V^{y_j} = \delta^{j1}$. Similarly, from (3.16)–(3.19) the Eulerian conserved density and fluxes are given by

$$F^0 = \rho u^k \hat{V}^{x^k} + \alpha t \mathcal{L}, \qquad F^j = \hat{V}^{x^k} (\rho u^k u^j + p \delta^{jk}) - \left(\alpha t u^j + \frac{\partial x^j}{\partial y_1} \right) \mathcal{L}, \tag{6.8}$$

where

$$\mathcal{L} = \frac{1}{2} \rho |\mathbf{u}|^2 - \frac{p}{\gamma - 1}, \tag{6.9}$$

is the Eulerian Lagrangian density (2.16) with $\mathbf{B} = 0$ and $\Phi = 0$.

6.2. *The projective symmetry* X_{14}

In this section we determine the conservation law associated with the projective symmetry X_{14} in gas dynamics with an adiabatic index $\gamma = (n+2)/n$, where n is the number of Cartesian space coordinates. The Eulerian form of the symmetry from (1.11) is

$$X_{14} = t x^\alpha \frac{\partial}{\partial x^\alpha} + (x^i - u^i t) \frac{\partial}{\partial u^i} - n t \rho \frac{\partial}{\partial \rho} - (n+2) t p \frac{\partial}{\partial p}. \tag{6.10}$$

By converting the Eulerian symmetry (6.10) to its Lagrangian equivalent we obtain the Lagrangian symmetry,

$$X_{14} = t^2 \frac{\partial}{\partial t} + t x^i \frac{\partial}{\partial x^i} + V^{\mathbf{x}_0} \cdot \nabla_0, \tag{6.11}$$

in $(t, \mathbf{x}, \mathbf{x}_0)$ Lagrange label space (see (3.34) *et seq*), and $V^{\mathbf{x}_0}$ is a fluid relabelling symmetry generator. In the present analysis we restrict our attention to the symmetry (6.11) in which $V^{\mathbf{x}_0} = 0$.

Proposition 6.3. *The projective symmetry (6.11) with* $V^{\mathbf{x}_0} = 0$ *is a divergence symmetry of action (3.1), where*

$$\mathcal{L}^0 = \frac{1}{2} \rho_0 |\mathbf{x}_t|^2 - \frac{p_0 J^{1-\gamma}}{\gamma - 1} \tag{6.12}$$

is the Lagrangian density, $\gamma = (n+2)/n$, $J = \det\left(\partial x^i / \partial x_0^j\right)$, $\rho_0 = \rho_0(\mathbf{x}_0)$, $p_0 = p_0(\mathbf{x}_0)$, $p = p_0 J^{-\gamma}$ *is the gas pressure and* $\mathbf{u} = \partial \mathbf{x}/\partial t$ *is the fluid velocity. Condition (3.9) for the symmetry (6.11) to be a divergence symmetry of the action reduces to*

$$\tilde{X} \mathcal{L}^0 + 2t \mathcal{L}^0 - \frac{\partial}{\partial t} \left(\frac{1}{2} \rho_0 |\mathbf{x}|^2 \right) = 0, \tag{6.13}$$

so that $\Lambda_0^0 = -\rho_0 |\mathbf{x}|^2/2$, $\Lambda_0^j = 0$ *in (3.9). Equation (6.13) shows that* X_{14} *is a divergence symmetry of the action and hence corresponds to a conservation law by Noether's theorem. This conservation law* $\partial I^0/\partial t + \partial I^j/\partial x_0^j = 0$ *has conserved density* I^0 *and flux* I^j *given by*

$$I^0 = t(x^k - u^k t) \rho_0 u^k + t^2 \mathcal{L}^0 - \tfrac{1}{2} \rho_0 |\mathbf{x}|^2, \qquad I^j = t(x^k - u^k t) p A_{kj}, \tag{6.14}$$

where $A_{kj} = \mathrm{cofac}(x_{kj})$ *and* $x_{kj} = \partial x^k / \partial x_0^j$.

Proof. The canonical symmetry generator $\hat{V}^{x^k}$, and the generators V^{x^k}, V^t, $V^{x^k_t}$, $V^{x_{ij}}$ in the extended derivative symmetry operator $\tilde{X}$ of (3.10) are given by

$$\begin{aligned} \hat{V}^{x^k} &= t(x^k - u^k t), \qquad V^t = t^2, \qquad V^{x^k} = tx^k, \\ V^{x^k_t} &= x^k - u^k t, \qquad V^{x_{ij}} = tx_{ij}. \end{aligned} \tag{6.15}$$

Evaluation of $\tilde{X}\mathcal{L}^0$ gives (6.13). The application of Noether's theorem (proposition 3.1) with $\mathbf{B} = 0$ gives the conserved density I^0 and flux components I^j in (6.14). This completes the proof. □

Comment 1

The symmetry (6.11) is a divergence symmetry of the action because (3.9) is satisfied and $\Lambda_0^\alpha \neq 0$.

Comment 2

The Eulerian form of the conservation equation, $\partial F^0/\partial t + \partial F^j/\partial x^j = 0$ follows from (3.14) *et seq*, where

$$F^0 = t(\mathbf{x} - \mathbf{u}t)\cdot\rho\mathbf{u} + t^2\mathcal{L} - \tfrac{1}{2}\rho|\mathbf{x}|^2, \tag{6.16}$$

$$\mathbf{F} = t(\mathbf{x} - \mathbf{u}t)\cdot(\rho\mathbf{u}\mathbf{u} + p\mathbf{I}) + \mathbf{u}\big(t^2\mathcal{L} - \tfrac{1}{2}\rho|\mathbf{x}|^2\big). \tag{6.17}$$

In (6.17), $\mathbf{F} = (F^1, F^2, F^3)$ is the spatial flux vector and $\mathbf{I}$ is the unit dyadic.

Comment 3

Ibragimov (1985) (section 25.2) derives the Eulerian conservation law $F_{0,t} + \nabla\cdot\mathbf{F} = 0$, where F_0 and $\mathbf{F}$ are given by (6.16) and (6.17), by applying the canonical Eulerian symmetry operator $\hat{X}_{14}$ twice to the Eulerian energy density and energy flux (i.e. to the Eulerian energy conservation equation). The essence of this approach is that a given Noether current C^α satisfying the conservation law, $D_\alpha C^\alpha = 0$, is mapped by the canonical Lie symmetry operator $\hat{X}$ onto the conservation law $D_\alpha F^\alpha = 0$, where $F^\alpha = \hat{X}C^\alpha$. This result is a consequence of the result $[\hat{X}, D_\alpha] = 0$ for the commutator of $\hat{X}$ with the total derivative operator D_α. In fact Ibragimov (1985) obtains two conservation laws associated with $\hat{X}_{14}$. It is not clear to the present authors how his first conservation law obtained by applying $\hat{X}_{14}$ once to the energy conservation law can be obtained by Noether's theorem, using the present approach.

The conservation law for the projective symmetries X_{14} for 2D MHD with an ignorable Cartesian coordinate z is discussed in appendix C.

7. Lie algebraic aspects

In this section we identify the Lie point symmetries of the Galilean group, and the projective symmetry X_{14} with the Lagrangian symmetries with $V^{\mathbf{x}_0} = 0$. Our aim is to determine the Lie brackets for the fluid relabelling symmetries for both gas dynamics (section 7.1) and MHD (section 7.2) and their relation to the generalized symmetry $X_{(a)}$ of (1.13) associated with the constant γ gas. In order for the Lie brackets to make sense, the two symmetry operators in the bracket $[\hat{X}_1, \hat{X}_2]$, namely $\hat{X}_1$ and $\hat{X}_2$ must correspond to the same differential equation system in Lagrange label space (i.e. the Lagrangian momentum equations (2.19) or the equivalent system of coupled nonlinear wave equations (A.1) for $x^i(\mathbf{x}_0, t)$). This implies that $\rho_0(\mathbf{x}_0)$, $S(\mathbf{x}_0)$ and $\mathbf{B}_0(\mathbf{x}_0)$ for the two symmetries are the same.

7.1. Gas dynamics

In this section we delineate the nature of the interaction between the Eulerian Lie point symmetries (1.1)–(1.11) in the case of pure gas dynamics with the fluid relabelling symmetries (e.g. Padhye and Morrison, (1996a, 1996b), Padhye (1998), Salmon (1982), Webb *et al* (2005b)). It turns out that the Lie symmetries (1.8)–(1.11) associated with an adiabatic equation of state with a constant adiabatic index γ have non-zero commutators with the fluid relabelling symmetries, but the commutators of the ten-parameter Galilean group have zero commutators with the fluid relabelling symmetries. This comes about because the adiabatic symmetries (1.8)–(1.11) depend on the equation of state and gas entropy $S = S(\mathbf{x}_0)$, and the entropy in turn depends on the Lagrange labels $\mathbf{x}_0$. On the other hand, the Galilean subgroup symmetries $\{X_i : 1 \leqslant i \leqslant 10\}$ (with $V^{\mathbf{x}_0} = 0$) do not depend on the Lagrange labels $\mathbf{x}_0$, and have zero commutators with the fluid relabelling symmetries.

7.1.1. The fluid relabelling sub-algebra. By searching for Lie symmetries of the form

$$x'^i = x^i, \qquad t' = t, \qquad x_0'^s = x_0^s + \epsilon V^{x_0^s}(\mathbf{x}_0), \tag{7.1}$$

which leave the action invariant, it was shown in section 3.3 that the symmetries must satisfy the Lie determining equations in $\mathbf{x}_0$ label space of the form (see also Padhye (1998), Webb *et al* (2005b))

$$\nabla_0 \cdot (\rho_0 V^{\mathbf{x}_0}) = 0, \qquad V^{\mathbf{x}_0} \cdot \nabla_0 S = 0. \tag{7.2}$$

Equations (7.2) arise from the Lie invariance condition (3.9) for the gas dynamic equations for the case $\Lambda_0^\alpha = 0$, corresponding to Lie transformations in which $V^{\mathbf{x}} = 0$ and $V^t = 0$, but $V^{\mathbf{x}_0} \neq 0$. Equations (7.2) have solutions of the form

$$V^{\mathbf{x}_0} = \frac{\nabla_0 S \times \nabla_0 \Omega}{\rho_0} \equiv \frac{\nabla_0 \times (S \nabla_0 \Omega)}{\rho_0}, \tag{7.3}$$

and are known as the fluid relabelling symmetries (e.g. Padhye and Morrison (1996a, 1996b), Padhye (1998), Salmon (1982), Webb *et al* (2005b)). Here $\Omega(\mathbf{x}_0)$ is an arbitrary, differentiable function of $\mathbf{x}_0$ (note also that $S = S(\mathbf{x}_0)$). This class of symmetries can be related to Ertel's theorem (i.e. the conservation of potential vorticity $\nabla \times \mathbf{u} \cdot \nabla S/\rho$ following the flow), via Noether's second theorem (e.g Padhye (1998)). The canonical Lie symmetry generator $\hat{V}^{x^k(\mathbf{x}_0,t)}$ associated with the symmetries (7.1) and (7.2) is given by the formula

$$\hat{V}^{\mathbf{x}} = V^{\mathbf{x}} - (V^t D_t + V^{\mathbf{x}_0} \cdot \nabla_0)\mathbf{x} \equiv -V^{\mathbf{x}_0} \cdot \nabla_0 \mathbf{x}. \tag{7.4}$$

From (7.3) and (7.4) we obtain

$$\hat{V}^{\mathbf{x}} = -\frac{\nabla S \times \nabla \Omega}{\rho} \equiv -\frac{\nabla \times (S \nabla \Omega)}{\rho}, \tag{7.5}$$

for the canonical symmetry operator corresponding to the relabelling symmetries (7.3) and (7.4). The derivation of (7.5) depends on the identity

$$\epsilon_{sab} x_{is} x_{ma} x_{nb} = \epsilon_{imn} J, \tag{7.6}$$

where $x_{ij} = \partial x^i/\partial x_0^j$ and $J = \det(x_{ij})$.

To determine the commutators of the fluid relabelling symmetry characteristic of the independent functions $\Omega_1(\mathbf{x}_0)$ and $\Omega_2(\mathbf{x}_0)$ we use the result

$$[\hat{X}(\hat{\eta}_1), \hat{X}(\hat{\eta}_2)] = \hat{X}(\hat{\eta}_3), \qquad \hat{\eta}_3^k = \hat{X}(\hat{\eta}_1)\hat{\eta}_2^k - \hat{X}(\hat{\eta}_2)\hat{\eta}_1^k, \tag{7.7}$$

for the commutator of two canonical Lie symmetries with generators $\hat{V}_1^{x^k} \equiv \hat{\eta}_1^k$ and $\hat{V}_2^{x^k} \equiv \hat{\eta}_2^k$ (e.g. Ibragimov (1985)).

Proposition 7.1. *The commutator of two fluid relabelling symmetries with canonical generators* $\hat{\eta}_j = -\nabla \times (S\nabla\Omega_j)/\rho$, $j = 1, 2$, *is given by*

$$[\hat{X}_{(r)}(\Omega_1), \hat{X}_{(r)}(\Omega_2)] = \hat{X}_{(r)}(\Omega_3), \tag{7.8}$$

where we use the notation $\hat{X}_{(r)}(\Omega_j) \equiv \hat{X}(\hat{\eta}_j)$ *to denote the relabelling symmetry associated with* Ω_j *and*

$$\Omega_3 = \frac{\nabla\Omega_1 \times \nabla\Omega_2 \cdot \nabla S}{\rho} \equiv \frac{\nabla_0\Omega_1 \times \nabla_0\Omega_2 \cdot \nabla_0 S}{\rho_0} = \frac{1}{\rho_0}\frac{\partial(\Omega_1, \Omega_2, S)}{\partial(x_0, y_0, z_0)}. \tag{7.9}$$

The symmetry associated with Ω_3 *has generator* $\hat{V}^{\mathbf{x}}(\Omega_3) = -\nabla \times (S\nabla\Omega_3)/\rho$. *The symmetry generator* $V^{\mathbf{x}_0}(\Omega_3)$ *is given by (7.3), but with* Ω *replaced by* Ω_3, *and hence is also a solution of the Lie determining equations (7.2).*

Proof. The proof follows directly from (7.7). □

Comment 1

The fluid relabelling symmetries form an infinite-dimensional Lie algebra, in which the commutator of two symmetries associated with Ω_1 and Ω_2 yields a symmetry of the same type associated with the function $\Omega_3(\mathbf{x}_0)$ given by (7.9). This result for the commutator of two fluid relabelling symmetries was also noted in Webb *et al* (2005b). Note that $\Omega_3 \neq 0$ provided Ω_1, Ω_2 and S are independent functions.

Comment 2

The Jacobi identity for the family of fluid relabelling symmetry operators is satisfied. The Jacobi identity follows by noting that

$$[X_1, X_2]f = X_1(X_2 f) - X_2(X_1 f), \tag{7.10}$$

defines the action of the Lie bracket on a function f (see e.g. Olver (1993), chapter 1, proposition 1.32). A direct verification of the Jacobi identity for the bracket (7.8) is given in appendix D.

7.1.2. Other commutators. The symmetry $X_{(a)}$ in (1.13) only applies if the entropy distribution $S(\mathbf{x}_0)$, density $\rho_0(\mathbf{x}_0)$ and symmetry generator $V^{\mathbf{x}_0}$ satisfy (6.1) and (6.2), and $\hat{V}^{\mathbf{x}}$ has the form (6.6). In the following analysis we assume that these conditions are satisfied.

It is straightforward to verify that

$$[\hat{X}_j, \hat{X}_{(r)}(\Omega)] = 0, \qquad 1 \leqslant j \leqslant 10. \tag{7.11}$$

In other words, the Galilean subgroup described by (1.1)–(1.5) or (1.12) has zero commutators with the fluid relabelling symmetries. This is because the generators of the Galilean group may be described by transformations in Lagrange label space of the form (3.2) in which there is no change in the Lagrange labels, i.e. $V^{\mathbf{x}_0} = 0$.

Proposition 7.2. *The commutator of the adiabatic symmetry (1.13),*

$$\hat{X}_{(a)} = \alpha\hat{X}_{11} + \beta\hat{X}_{12} + \delta\hat{X}_{13}, \tag{7.12}$$

with the canonical fluid relabelling symmetry,

$$\hat{X}_{(r)}(\Omega) = -\frac{\nabla \times (S\nabla\Omega)}{\rho} \cdot \nabla + \cdots, \tag{7.13}$$

is given by

$$[\hat{X}_{(a)}, \hat{X}_{(r)}(\Omega)] = \hat{X}_{(r)}(\Omega_a), \tag{7.14}$$

where

$$\Omega_a = -\delta_2\Omega + V_a^{\mathbf{x}_0}\cdot\nabla_0\Omega \equiv -\delta_2\Omega + \frac{\partial\Omega}{\partial y_1}, \tag{7.15}$$

and the entropy $S(\mathbf{x}_0)$ and density ρ_0 have the form (6.4) required for the 'adiabatic' symmetry (1.13). Thus, the commutator of the adiabatic symmetry $\hat{X}_{(a)}$ with the fluid relabelling symmetry $\hat{X}_{(r)}(\Omega)$ gives rise to a further relabelling symmetry $\hat{X}_{(r)}(\Omega_a)$, where $\Omega_a = -\delta_2\Omega + V_a^{\mathbf{x}_0}\cdot\nabla_0\Omega$ and $V_a^{\mathbf{x}_0} = \partial\mathbf{x}_0/\partial y_1$ describes transformations associated with $X_{(a)}$ in $\mathbf{x}_0$ label space.

Proof. To prove (7.13), we first note that

$$\nabla_0\bar{S} = 2\delta_1\omega^1 + D_2(y_2, y_3)\omega^2 + D_3(y_2, y_3)\omega^3, \tag{7.16}$$

where $\omega^j = \nabla_0 y_j$ are the covariant base vectors introduced following (4.39) and $D_j \equiv \partial D/\partial y_j$ $(j = 2, 3)$. The canonical symmetry generator $\hat{V}_{(r)}^{\mathbf{x}}$ for the relabelling symmetry can be written in the form

$$\hat{V}_{(r)}^{\mathbf{x}} = -V^{\mathbf{x}_0}\cdot\nabla_0\mathbf{x} = -V_{(r)}^{y_s}\frac{\partial\mathbf{x}}{\partial y_s}, \tag{7.17}$$

where

$$V_{(r)}^{y_s}(\mathbf{y}) = \frac{(\tilde{\nabla}S\times\tilde{\nabla}\Omega)^s}{\tilde{\rho}_0}, \qquad \tilde{\nabla}_j = \frac{\partial}{\partial y_j}, \qquad \tilde{\rho}_0 = \sqrt{g}\rho_0 = \Phi(y_2, y_3)\exp(\delta_2 y_1). \tag{7.18}$$

From (7.16)–(7.18), we obtain

$$\hat{V}_{(r)}^{x^k} = F^s(\mathbf{y})\frac{\partial x^k}{\partial y_s}, \qquad \mathbf{F}(\mathbf{y}) = -\frac{1}{\tilde{\rho}_0}\left(D_2\Omega_3 - D_3\Omega_2, D_3\Omega_1 - 2\delta_1\Omega_3, 2\delta_1\Omega_2 - D_2\Omega_3\right)^T. \tag{7.19}$$

Similarly,

$$\hat{V}_{(a)}^{x^k} = V^{x^k} - \left(V^t D_t + V^{y_s}D_{y_s}\right)x^k \equiv \beta x^k - \alpha t u^k - \frac{\partial x^k}{\partial y_1}. \tag{7.20}$$

Using $\hat{\eta}_1 = \hat{V}_{(a)}^{\mathbf{x}}$ and $\hat{\eta}_2 = \hat{V}_{(r)}^{\mathbf{x}}(\Omega)$ in (7.7) now gives (7.14) for the commutator $[\hat{X}_{(a)}, \hat{X}_{(r)}(\Omega)]$. This completes the proof. □

7.2. MHD

In this section we investigate the nature of the fluid relabelling symmetries and the symmetries associated with the symmetry $X_{(a)}$ of (1.13).

7.2.1. The MHD fluid relabelling sub-algebra. The MHD fluid relabelling symmetries are infinitesimal Lie transformations of the form $x'^i = x^i, t' = t, x_0'^i = x_0^i + \epsilon V^{x_0^i}$, which leave the MHD action invariant. This class of symmetries has Lie generators $V^{\mathbf{x}_0}$ and the related Lagrange space variables $\rho_0(\mathbf{x}_0)$, $\bar{S}(\mathbf{x}_0)$, and $\mathbf{B}_0(\mathbf{x}_0)$ satisfying the Lie determining equations (3.24), namely,

$$\nabla_0\cdot(\rho_0 V^{\mathbf{x}_0}) = 0, \qquad V^{\mathbf{x}_0}\cdot\nabla_0\bar{S} = 0, \qquad D_t(V^{\mathbf{x}_0}) = 0, \qquad \nabla_0\times(V^{\mathbf{x}_0}\times\mathbf{B}_0) = 0, \qquad \nabla_0\cdot\mathbf{B}_0 = 0. \tag{7.21}$$

The solution of the Lie determining equations (7.21) depends on whether (i) $V^{\mathbf{x}_0}\times\mathbf{B}_0=\nabla_0\psi\neq 0$, or (ii) $V^{\mathbf{x}_0}\parallel\mathbf{B}_0$. There is also a simplification of the solution of the equations if (iii) the gas is barotropic, i.e. the equation of state does not depend on the entropy and $\varepsilon=\varepsilon(\rho)$. The solution of (7.21) for these cases is discussed below.

Case 1. $V^{x_0}\times\mathbf{B}_0=\nabla_0\psi$

In this case, (7.21) has solutions

$$V^{\mathbf{x}_0}=\mathbf{e}_1,\qquad \mathbf{B}_0=\rho_0\mathbf{e}_2,\qquad \rho_0=\frac{1}{\sqrt{g}},\qquad S=S(\chi,\psi),\tag{7.22}$$

where

$$\mathbf{e}_1=\frac{\partial\mathbf{x}_0}{\partial\phi},\qquad \mathbf{e}_2=\frac{\partial\mathbf{x}_0}{\partial\chi},\qquad \mathbf{e}_3=\frac{\partial\mathbf{x}_0}{\partial\psi},\tag{7.23}$$

$$g=\det(g_{ij}),\qquad g_{ij}=\mathbf{e}_i\cdot\mathbf{e}_j,\tag{7.24}$$

(e.g. Webb *et al* (2005b)). Solutions (7.22)–(7.24) are obtained using the same ideas used to solve Lie determining equations in section 4.2. In the present analysis we use the notation $(\phi,\chi,\psi)\equiv(y_1,y_2,y_3)$ to denote the potentials in $\mathbf{x}_0$ label space characterizing the solutions. Note that the vector fields,

$$X_1=V^{\mathbf{x}_0}\cdot\nabla_0\equiv\frac{\partial}{\partial\phi}\qquad\text{and}\qquad X_2=\frac{\mathbf{B}_0}{\rho_0}\cdot\nabla_0\equiv\frac{\partial}{\partial\chi},\tag{7.25}$$

satisfy the Abelian Lie algebra $[X_1,X_2]=0$. The geometrical formalism of section 4.2.1 also applies in the present case. For example $\omega^1=\nabla_0\phi$, $\omega^2=\nabla_0\chi$ and $\omega^3=\nabla_0\psi$ are the covariant dual basis vectors corresponding to the contravariant base vectors $\{\mathbf{e}_j: j=1,2,3\}$ satisfying (4.40).

Case 2. $V^{\mathbf{x}_0}\parallel\mathbf{B}_0$

In this case, (7.21) has solutions for $V^{\mathbf{x}_0}$ and $\mathbf{B}_0$ of the form

$$\begin{aligned}V^{\mathbf{x}_0}&=-\frac{\zeta(\mathbf{x}_0)}{\rho_0}\mathbf{B}_0=-\frac{\nabla_0\times(\Omega\nabla_0 S)}{\rho_0},\\ \mathbf{B}_0&=\frac{\nabla_0\Omega\times\nabla_0 S}{\zeta}\equiv\nabla_0\Phi\times\nabla_0 S,\end{aligned}\tag{7.26}$$

where $\zeta(\mathbf{x}_0)$, $\Phi(S,\zeta)$ and $\Omega(S,\zeta)$ satisfy the equation

$$\Omega(S,\zeta)=\int^{\zeta}\zeta'\frac{\partial\Phi(S,\zeta')}{\partial\zeta'}\,\mathrm{d}\zeta'.\tag{7.27}$$

Comment

The derivation of (7.26) and (7.27) follows by noting that $\nabla_0\cdot(\rho_0V^{\mathbf{x}_0})=0$, $V^{\mathbf{x}_0}\cdot\nabla_0\bar{S}=0$ and $\nabla_0\cdot\mathbf{B}_0=0$ require $\mathbf{B}_0\cdot\nabla_0\zeta=0$, $\mathbf{B}_0\cdot\nabla_0\bar{S}=0$ and $\nabla_0\cdot\mathbf{B}_0=0$. The equations $\mathbf{B}_0\cdot\nabla_0\bar{S}=0$ and $\nabla_0\cdot\mathbf{B}_0=0$ are satisfied by $\mathbf{B}_0=\nabla_0\Phi\times\nabla_0\bar{S}$. The equation $\mathbf{B}_0\cdot\nabla_0\zeta=0$ reduces to the Jacobian equation $\partial(\Phi,S,\zeta)/\partial(x_0,y_0,z_0)=0$, from which it follows that $\Phi=\Phi(S,\zeta)$. The function $\Omega(S,\zeta)$ in (7.27) allows the solutions for $V^{\mathbf{x}_0}$ and $\mathbf{B}_0$ to be expressed in the form (7.26).

Case 3. Barotropic MHD

For barotropic MHD, there is no constraint associated with the entropy, since the equation of state $\varepsilon=\varepsilon(\rho)$ does not involve the entropy. This is most easily seen from condition

(3.9) (see Webb *et al* (2005b) equation (5.83)). Hence the entropy in this case is an arbitrary function of $\mathbf{x}_0$. There are two cases to be considered according as $V^{\mathbf{x}_0} \parallel \mathbf{B}_0$ or not. For the Case $V^{\mathbf{x}_0} \parallel \mathbf{B}_0$ the Lie determining equations admit solutions

$$V^{\mathbf{x}_0} = -\psi(\mathbf{x}_0)\frac{\mathbf{B}_0}{\rho_0} \qquad \text{and} \qquad \mathbf{B}_0 = \nabla_0\Phi \times \nabla_0\psi, \tag{7.28}$$

where $\psi(\mathbf{x}_0)$ and $\Phi(\mathbf{x}_0)$ are arbitrary functions of $\mathbf{x}_0$. We will not consider this case further in the present paper.

Proposition 7.3. *The commutator of two non-field aligned, fluid relabelling symmetries of the form (7.21) in which $V^{\mathbf{x}_0}$ are not parallel to $\mathbf{B}_0$, with Lie generators*

$$V_j^{\mathbf{x}_0} = \frac{\nabla_0\chi_j \times \nabla_0\psi_j}{\rho_0}, \qquad \hat{V}_j^{\mathbf{x}} = -V_j^{\mathbf{x}_0}\cdot\nabla_0\mathbf{x}, \quad j = 1, 2, \tag{7.29}$$

with the same $\rho_0(\mathbf{x}_0)$, $S(\mathbf{x}_0)$ and $\mathbf{B}_0(\mathbf{x}_0)$ distributions have zero commutator:

$$\left[\hat{X}\left(\hat{V}_1^{\mathbf{x}}\right), \hat{X}\left(\hat{V}_2^{\mathbf{x}}\right)\right] = \hat{X}\left(\hat{V}_3^{\mathbf{x}}\right). \tag{7.30}$$

In other words, $\hat{V}_3^{\mathbf{x}} = 0$. In order for the two symmetries (7.29) to correspond to the same $\rho_0(\mathbf{x}_0)$, $S(\mathbf{x}_0)$ and $\mathbf{B}_0(\mathbf{x}_0)$ requires $\chi_1 = \chi_2$, $\psi_2 = f(\psi_1, \chi_1)$ for some differentiable function f, and

$$V_2^{\mathbf{x}_0} = f_{\psi_1}(\psi_1, \chi_1)V_1^{\mathbf{x}_0}. \tag{7.31}$$

Proof. From (7.22)–(7.24), $\mathbf{B}_0 = \rho_0\mathbf{e}_2$ and $\rho_0(\mathbf{x}_0)$ for the two symmetries are the same if $\chi_1 = \chi_2$ and $\rho_0(\mathbf{x}_0) = 1/\sqrt{g}$ for the two symmetries. The entropy $S(\mathbf{x}_0) = S_1(\chi_1, \psi_1) = S_2(\chi_2, \psi_2)$ and $\chi_1 = \chi_2$ requires $\psi_2 = f(\psi_1, \chi_1)$. These conditions then imply $V_j^{\mathbf{x}_0}$ for $j = 1, 2$ and are related by (7.31).

Using (7.7), $\hat{V}_3^{\mathbf{x}}$ in (7.30) may be expressed in the form

$$\begin{aligned} &\hat{V}_3^{\mathbf{x}} = -V_3^{\mathbf{x}_0}\cdot\nabla_0\mathbf{x}, \qquad V_3^{\mathbf{x}_0} = -\frac{\nabla_0\times(\Phi\nabla_0 S)}{\rho_0}, \\ &\Phi = \chi_{1,\bar{S}}\chi_{2,\bar{S}}\frac{\nabla_0\psi_1\times\nabla_0\psi_2\cdot\nabla_0\bar{S}}{\rho_0}, \end{aligned} \tag{7.32}$$

where $\chi_{j,\bar{S}} = \partial\chi_j/\partial\bar{S}$ $(j = 1, 2)$. Using the fact that we can express $\bar{S}$ in terms of either (χ_1, ψ_1) or (χ_2, ψ_2), we find

$$\Phi = \chi_{2,\bar{S}}\frac{\nabla_0\chi_1\times\nabla_0\psi_1\cdot\nabla_0\psi_2}{\rho_0} \equiv \chi_{2,\bar{S}}\frac{1}{\rho_0}\frac{\partial(\chi_1, \psi_1, \psi_2)}{\partial(x_0.y_0, z_0)} \equiv 0. \tag{7.33}$$

In (7.33), $\Phi \equiv 0$ because ψ_2, χ_1 and ψ_1 are not independent functions (i.e. $\psi_2 = f(\chi_1, \psi_1)$ for some f), for the common entropy distribution used for the symmetries. Hence, $V_3^{\mathbf{x}_0} = \hat{V}_3^{\mathbf{x}} = 0$. This completes the proof. □

Proposition 7.4. *Two field-aligned ($V^{\mathbf{x}_0} \parallel \mathbf{B}_0$) fluid relabelling symmetries, with Lie generators:*

$$V_j^{\mathbf{x}_0} = -\frac{\nabla_0\times(\Omega_j\nabla_0 S)}{\rho_0}, \qquad j = 1, 2, \tag{7.34}$$

with the same $\rho_0(\mathbf{x}_0)$, $S(\mathbf{x}_0)$ and $\mathbf{B}_0$ can only differ at most in the functions $\zeta_j(\mathbf{x}_0)$, i.e., $V_j^{\mathbf{x}_0} = -\zeta_j(\mathbf{x}_0)\mathbf{B}_0/\rho_0$, $(j = 1, 2)$. Using the notation $\Omega_j = \Omega(S, \zeta_j(\mathbf{x}_0))$ $(j = 1, 2)$ the symmetries (7.34) satisfy commutation relations (7.30), where

$$\hat{V}_j^{\mathbf{x}} = -V_j^{\mathbf{x}_0}\cdot\nabla_0\mathbf{x} \equiv \frac{\nabla\times(\Omega_j\nabla S)}{\rho}, \qquad j = 1, 2, 3, \tag{7.35}$$

and

$$\Omega_3 = \frac{\nabla_0\Omega_1 \times \nabla_0\Omega_2 \cdot \nabla_0 S}{\rho_0} \equiv \frac{\nabla\Omega_1 \times \nabla\Omega_2 \cdot \nabla S}{\rho}. \tag{7.36}$$

Remark 1. The commutation relations are similar to the pure fluid dynamical case described in (7.8) *et seq*.

Remark 2. Note that $\Omega_3 \neq 0$ if $\partial(S, \Omega_1, \Omega_2)/\partial(x_0, y_0, z_0) \neq 0$.

Proposition 7.5. *The commutator of a non-field aligned MHD fluid relabelling symmetry, with generator*

$$\hat{V}_1^{\mathbf{x}} = -V_1^{\mathbf{x}_0} \cdot \nabla_0 \mathbf{x}, \qquad V_1^{\mathbf{x}_0} = \frac{\nabla_0\chi \times \nabla_0\psi}{\rho_0}, \tag{7.37}$$

where the entropy $S = S(\chi, \psi)$ and $\rho_0 = 1/\sqrt{g}$ (cf (7.22) et seq), and a field aligned MHD fluid relabelling symmetry, with canonical generator,

$$\hat{V}_2^{\mathbf{x}} = -V_2^{\mathbf{x}_0} \cdot \nabla_0 \mathbf{x}, \qquad V_2^{\mathbf{x}_0} = \frac{\nabla_0 S \times \nabla_0\Omega}{\rho_0}, \tag{7.38}$$

have commutators of the form (7.30) with

$$\begin{aligned} &\hat{V}_3^{\mathbf{x}} = -V_3^{\mathbf{x}_0} \cdot \nabla_0 \mathbf{x}, \qquad V_3^{\mathbf{x}_0} = \frac{\nabla_0 S \times \nabla_0\Phi}{\rho_0}, \\ &\Phi = \frac{\partial\chi}{\partial S}\frac{\nabla_0\psi \times \nabla_0\Omega \cdot \nabla_0 S}{\rho_0} \equiv \frac{\nabla_0\psi \times \nabla_0\Omega \cdot \nabla_0\chi}{\rho_0}, \end{aligned} \tag{7.39}$$

and $\partial\chi/\partial S \equiv \partial\chi(S, \psi)/\partial S$.

Remark 1. The canonical symmetry generators $\hat{V}_j^{\mathbf{x}}$ $(j = 1, 2, 3)$ can be written in the form

$$\hat{V}_1^{\mathbf{x}} = -\frac{\nabla \times (\chi\nabla\psi)}{\rho}, \qquad \hat{V}_2^{\mathbf{x}} = -\frac{\nabla \times (S\nabla\Omega)}{\rho}, \qquad \hat{V}_3^{\mathbf{x}} = -\frac{\nabla \times (S\nabla\Phi)}{\rho}. \tag{7.40}$$

Thus, $\hat{V}_3^{\mathbf{x}}$ has the form of a pure fluid relabelling symmetry or a field-aligned symmetry with potential Φ.

Remark 2. If $\partial(\Omega, \psi, \chi)/\partial(x_0, y_0, z_0) = 0$ then $\Phi = 0$. This occurs if $\Omega = \Omega(\chi, \psi)$.

An example of the commutator of a fluid relabelling symmetry $\hat{X}_{(r)}$ for which $\mathbf{B}_0 \neq 0$, with an adiabatic symmetry $\hat{X}_{(a)}$ with the same $\rho_0(\mathbf{x}_0)$, $\mathbf{B}_0(\mathbf{x}_0)$ and $S(\mathbf{x}_0)$ is given below.

Proposition 7.6. *The fluid relabeling symmetry (7.22) with*

$$\begin{aligned} &V_{(r)}^{\mathbf{x}_0} = \mathbf{e}_1, \qquad \mathbf{B}_0 = \rho_0\mathbf{e}_2, \qquad \rho_0 = \frac{1}{\sqrt{g}}, \\ &S(\mathbf{x}_0) = 2\frac{\delta_1}{\delta_2}\ln(y_3) + D(y_2), \end{aligned} \tag{7.41}$$

has zero commutator

$$[\hat{X}_{(a)}, \hat{X}_{(r)}] = 0, \tag{7.42}$$

with the adiabatic symmetry $\hat{X}_{(a)}$ with symmetry generators

$$V_{(a)}^{\mathbf{x}} = \beta\mathbf{x}, \qquad V_{(a)}^{t} = \alpha t, \qquad V_{(a)}^{\mathbf{x}_0} = \delta_2 y_3 \mathbf{e}_3, \tag{7.43}$$

where both symmetries have the same $\rho_0(\mathbf{x}_0)$, $\mathbf{B}_0(\mathbf{x}_0)$ and $S(\mathbf{x}_0)$ distributions given in (7.41).

Proof. The main problem is to determine ρ_0, $\mathbf{B}_0$ and S distributions that are compatible with both symmetries. Using solution (4.43) for $\delta_4 = \delta + 2\alpha - \beta = 0$, with potentials $y_1' = \ln(y_3)/\delta_2$, $y_2' = y_2$, $y_3' = -y_3$, $F(y_3') = 1$ results in the base vectors $\mathbf{e}_1' = \delta_2 y_3 \mathbf{e}_3$, $\mathbf{e}_2' = \mathbf{e}_2$ and $\mathbf{e}_3' = -\mathbf{e}_1$ and $g' = \theta^2 g$, where $\theta = \exp(-\delta_2 y_3)/\delta_2$ leads to the distributions in (7.41). One can check directly that $V_{(a)}^{\mathbf{x}_0}$ in (7.41) satisfies the Lie determining equations (4.4)–(4.7) for the case $\delta_4 = 0$. Note that

$$\hat{V}_{(a)}^{\mathbf{x}} = \beta\mathbf{x} - \alpha t\mathbf{u} - \delta_2 y_3 \frac{\partial\mathbf{x}}{\partial y_3}, \qquad \hat{V}_{(r)}^{\mathbf{x}} = -\frac{\partial\mathbf{x}}{\partial y_1}, \tag{7.44}$$

are the canonical symmetry generators involved. □

This completes our discussion of the Lie symmetries $X_{(a)}$ and $X_{(r)}$ for the MHD case. More complicated examples could be constructed.

8. Summary and concluding remarks

In this paper we have explored the role of the Lagrangian map for Lie symmetries in MHD and gas dynamics. The analysis made use of the Lagrangian action principle for MHD and fluid dynamics developed by Newcomb (1962) based on the Lagrangian map, in which the Eulerian position coordinate of the fluid element, $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)$ is regarded as a function of the Lagrangian fluid labels $\mathbf{x}_0$ and time t. The Lagrangian map is the solution of the differential equation $\mathrm{d}\mathbf{x}/\mathrm{d}t = \mathbf{u}(\mathbf{x}, t)$, where $\mathbf{x} = \mathbf{x}_0$ at time $t = 0$.

After establishing the basic Lagrangian action and variational principle for MHD (section 2), the Lagrangian and Eulerian conservation laws associated with a given variational or divergence symmetry of the action (Noether's theorem) were obtained (section 3). This formulation of Noether's theorem involved infinitesimal Lie transformations of the form

$$x'^i = x^i + \epsilon V^{x^i}, \qquad t' = t + \epsilon V^t, \qquad x_0'^i = x_0^i + \epsilon V^{x_0^i}, \tag{8.1}$$

that leave the action invariant, where $\mathbf{x} = \mathbf{x}(\mathbf{x}_0, t)$ is given by the Lagrange map. Central to the analysis is the Lie invariance condition (3.9) for the symmetry to be a variational or divergence symmetry of the action. If condition (3.9) is satisfied then the associated conservation law follows from Noether's theorem. The fluid relabelling symmetries in this analysis correspond to Lie transformations of the form (8.1) in which $\mathbf{x}$ and t are fixed, but transformations in the fluid labels $\mathbf{x}_0$ are allowed (i.e. $V^t = V^{\mathbf{x}} = 0$ but $V^{\mathbf{x}_0}$ can be non-zero). The Lie equations for the fluid relabelling symmetries (3.24) then follow from condition (3.9) that the transformation leaves the action invariant. These equations are analogous to steady MHD equations in which $V^{\mathbf{x}_0}$ plays the role of the fluid velocity. These equations are the analogues of: the steady mass continuity equation, the entropy advection equation, and the steady version of Faraday's law and Gauss's equation $\nabla \cdot \mathbf{B}_0 = 0$. This is the approach to fluid relabelling symmetries used by Padhye and Morrison (1996a, 1996b), Padhye (1998) and Webb *et al* (2005b).

By using the Lie extension formulae for the transformation of the derivatives of the dependent variables (i.e. $\partial\mathbf{x}/\partial t$ and $\partial x^i/\partial x_0^j$) coupled with the transformation formulae of the physical variables from the Lagrange label point $\mathbf{x}_0$ at time $t = 0$ to the Eulerian position $\mathbf{x}$ at time t, allows one to transform a given Lie symmetry from its Eulerian form to its equivalent Lagrange label space form. Applying this idea to the Eulerian Lie point symmetries of the Galilei group (1.1)–(1.5), one obtains Lagrangian symmetry operators, that have the same infinitesimal generators V^t and $V^{\mathbf{x}}$ as the Eulerian symmetries, but solutions of the Lie determining equations allow $V^{\mathbf{x}_0} \neq 0$ that satisfy the fluid relabelling equations (3.24). In other words, there is an infinite class of symmetries in Lagrange label space that map onto the given Eulerian Lie point symmetry of the Galilei group. Similarly, the projective symmetry X_{14} for gas dynamics, with an adiabatic index $\gamma = (n+2)/n$ of (1.11), has symmetries with the same $V^{\mathbf{x}}$ and V^t as the Eulerian symmetry operator, but with $V^{\mathbf{x}_0}$ corresponding to a fluid relabelling symmetry.

By converting the scaling symmetry $X_{(a)}$ of (1.13), associated with a constant adiabatic index gas, to its Lagrangian form, one finds that the infinitesimal symmetry generator $V^{\mathbf{x}_0}$ in (8.1) must now satisfy a modified form of the fluid relabelling determining equations with non-zero source terms (equations (4.4)–(4.7)). Analysis of (4.4)–(4.7) shows that $X_{(a)}$ corresponds to a variational symmetry of the action if $\alpha + 2\delta + n\beta = 0$, which is condition (1.14). The solution of the Lie equations (4.4)–(4.7) was carried out by demonstrating that the vector fields $X_1 = \mathbf{b}_0 \cdot \nabla_0$, where $\mathbf{b}_0 = \mathbf{B}_0/\rho_0$ and $X_2 = V^{\mathbf{x}_0} \cdot \nabla_0$, form a two-dimensional Lie algebra, with commutation relation $[X_1, X_2] = \delta_4 X_1$, where $\delta_4 = \delta + 2\alpha - \beta$. A representation for the Lie algebra, coupled with Frobenius's integrability theorem leads to solutions of the Lie determining equations (4.4)–(4.7) depending on the parameters. The solutions were used in section 5 to determine both Lagrangian and Eulerian MHD conservation laws in the case $\alpha + 2\delta + n\beta = 0$. A similar analysis in section 6 for the symmetry $X_{(a)}$ gives the corresponding conservation laws for the gas dynamical model. The conservation laws associated with the symmetry $X_{(a)}$ only apply for specific initial data for the physical variables. We also use Noether's theorem in section 6 to derive the conservation law associated with the projective symmetry X_{14} for an ideal gas with an adiabatic index $\gamma = (n+2)/n$, where n is the number of Cartesian space dimensions. The conservation law for the projective symmetry for 2D ideal MHD, with an ignorable coordinate z, with a transverse magnetic field $\mathbf{B} = (0, 0, B)^t$ (see also Fuchs and Richter (1987)), is discussed in appendix C.

In section 7, we investigated the Lie algebraic commutation relations for the fluid relabelling symmetries and the symmetries $X_{(a)}$ of (1.13). Both the fluid relabelling symmetries and $X_{(a)}$ are determined in part by the equation of state of the gas (i.e. the entropy distribution). As a consequence the Lagrangian, canonical symmetry operators for these symmetries have non-trivial Lie algebraic commutation relations.

We have restricted our attention to symmetries in Lagrange label space that are essentially Lie point symmetries (i.e. they do not depend on the derivatives of $\mathbf{x}(\mathbf{x}_0, t)$). The role of non-point symmetries in Lagrange label space is not clear from the present analysis, and requires further work. It is of interest to investigate the role of symmetries in Lagrangian mean-wave theories of turbulence, as developed for example by Holm (1999), and on the Lagrangian averaged Euler Poincaré equations (LAEP) equations. The fluid relabelling symmetries can presumably be investigated by using Cartan's method of moving frames, and the algebra of exterior differential forms (e.g. Harrison and Estabrook (1971), Olver and Pohjanpelto (2005)), and the multi-symplectic approach (e.g Bridges *et al* (2005) and Hydon (2005)). The fluid relabelling symmetries are presumably related to recent work by Gibbon *et al* (2006) on vorticity dynamics in incompressible 3D fluid dynamics, where the vortex dynamics are described using a quaternion representation for the Lagrangian advection and stretching of the vortex tubes.

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Appendix A

In this appendix we reduce the Lagrangian momentum equation (2.19) to a system of three coupled nonlinear MHD wave equations for $x^i(\mathbf{x}_0, t)$ $(1 \leqslant i \leqslant 3)$. We also discuss the characteristic manifold of the system of coupled PDEs for the $x^i(\mathbf{x}_0, t)$.

Using the Lagrangian map equations (2.7)–(2.9) for ρ and $\mathbf{B}$ and noting that $p = p(\rho, S)$, (2.19) reduces the coupled wave system:

$$A_j^{i\alpha\beta} x^j_{\alpha\beta} + R^i = 0, \tag{A.1}$$

where $x^j_{\alpha\beta} = \partial^2 x^j/\partial x_0^\alpha \partial x_0^\beta$ $(\alpha, \beta = 0, 1, 2, 3, 1 \leqslant j \leqslant 3$, i.e. Greek indices assume the values 0, 1, 2, 3 and Latin indices assume the values 1, 2, 3) and $(t, x, y, z) \equiv (x^0, x^1, x^2, x^3)$. In (A.1),

$$A_j^{i\alpha\beta} = \delta^{ij}[\delta^{\alpha 0}\delta^{\beta 0} - \delta^{\alpha p}\delta^{\beta q} b^s b^k y_{pk} y_{qs}] \\ + \delta^{\alpha p}\delta^{\beta q}[-(a^2+b^2) y_{qi} y_{pj} + b^j b^s y_{qi} y_{ps} + b^s b^i y_{pj} y_{qs}], \tag{A.2}$$

$$R^i = \frac{B^r}{\mu\rho_0}\frac{\partial B_0^s}{\partial x_0^p}(y_{pi}x_{rs} - x_{is}y_{pr}) + \frac{A_{ij}}{\rho_0}\left(\frac{a^2}{J} + \frac{\partial p}{\partial S}\frac{\partial S}{\partial x_0^j}\right) + \frac{\partial \Phi}{\partial x^i}. \tag{A.3}$$

Here $\mathbf{b} = \mathbf{B}/\sqrt{\mu\rho}$ is the local Alfvén velocity and $a = (\partial p/\partial \rho)^{1/2}$ is the adiabatic sound speed of the gas. The characteristic manifolds of the partial differential equation system (A.1) are defined as manifolds $\phi(\mathbf{x}, t) = \text{const}$ on which the Cauchy, initial value problem does not have a unique solution. The characteristic manifolds of (A.1) are given by the solutions of the determinantal equation:

$$\det(\tilde{\mathbf{A}}) = 0, \qquad \text{where} \quad \tilde{A}^i_j = A_j^{i\alpha\beta}\phi_{,\alpha}\phi_{,\beta} \tag{A.4}$$

and $\phi_{,\alpha} \equiv \partial\phi/\partial x_0^\alpha$. The matrix $\tilde{\mathbf{A}}$ can be written in the form

$$\tilde{A}^i_j = [\omega'^2 - (\mathbf{b}\cdot\mathbf{k})^2]\delta^{ij} - (a^2+b^2)k^i k^j + (\mathbf{b}\cdot b\mathbf{k})(b^i k^j + b^j k^i), \tag{A.5}$$

where

$$\omega' = -\frac{\partial\phi(\mathbf{x}_0, t)}{\partial t} \equiv -(\phi_t + \mathbf{u}\cdot\nabla\phi), \quad \mathbf{k} = \nabla\phi, \tag{A.6}$$

(ϕ_t denotes the time derivative of ϕ keeping $\mathbf{x}$ constant) define the wave frequency ω' in the fluid frame and wave number $\mathbf{k}$. The determinant of $\tilde{\mathbf{A}}$ is

$$\det(\tilde{\mathbf{A}}) = [\omega'^2 - (\mathbf{b}\cdot\mathbf{k})^2][\omega'^4 - (a^2+b^2)\omega'^2 k^2 + a^2 k^2 (\mathbf{b}\cdot\mathbf{k})^2] = 0, \tag{A.7}$$

(cf Webb *et al* (2005a)). The first factor in (A.7) corresponds to the Alfvén wave modes, and the second factor in square brackets corresponds to the fast and slow magnetosonic modes respectively.

Appendix B

In this Appendix we indicate the origin of the identity (3.13). In the derivation of Noether's first theorem (e.g. Bluman and Kumei (1989)) for the case in which action (3.1) is invariant under the canonical Lie transformation:

$$x'^k = x^k + \epsilon\hat{V}^{x^k}, \quad t' = t, \quad x_0'^j = x_0^j, \tag{B.1}$$

where

$$\hat{V}^{x^k} = V^{x^k} - \left(V^t D_t + V^{x_0^j} D_{x_0^j}\right) x^k(\mathbf{x}_0, t), \tag{B.2}$$

the variation of the action is given by

$$\Delta A = \lim_{\epsilon \to 0} \frac{A[\mathbf{x} + \epsilon \hat{V}^{\mathbf{x}}] - A[\mathbf{x}]}{\epsilon} = \int_{\mathcal{R}} \mathrm{d}^3 x_0 \, \mathrm{d}t \, \hat{X} \mathcal{L}^0, \tag{B.3}$$

where $\hat{X}$ is the canonical Lie symmetry operator (3.12). However, by using integration by parts one can also obtain (e.g. Bluman and Kumei (1989))

$$\Delta A = \int_{\mathcal{R}} \mathrm{d}^3 x_0 \, \mathrm{d}t \left[\hat{V}^{x^k} E_{x^k}(\mathcal{L}^0) + D_\alpha W^\alpha \right], \tag{B.4}$$

where $E_{x^k}(\mathcal{L}^0) \equiv \delta A / \delta x^k$ is the Euler operator for x^k and

$$W^\alpha = \hat{V}^{x^k} \frac{\partial \mathcal{L}^0}{\partial x^k_\alpha} \quad (\alpha = 0, 1, 2, 3) \tag{B.5}$$

are surface vector terms. Since the integration region $\mathcal{R}$ is arbitrary in (B.3) and (B.4) we deduce

$$\hat{X} \mathcal{L}^0 = \hat{V}^{x^k} E_{x^k}(\mathcal{L}^0) + D_\alpha W^\alpha. \tag{B.6}$$

Adding $D_\alpha\left(\mathcal{L}^0 V^\alpha + \Lambda_0^\alpha\right)$ to both sides of (B.6) and noting $\tilde{X} = \hat{X} + V^\alpha D_\alpha$ (see (3.11)), we obtain identity (3.13). The above argument is essentially a condensed form of that given by Bluman and Kumei (1989) in a more general proof of Noether's theorem. In the above analysis we assumed that $\mathcal{L}^0$ depended at most on first order derivatives of the dependent variables $x^k(\mathbf{x}_0, t)$. Bluman and Kumei consider the more general case where $\mathcal{L}^0$ can depend on derivatives of any order of the dependent variables.

One can verify (3.13) directly by noting that

$$\begin{aligned} D_\alpha(W^\alpha + V^\alpha \mathcal{L}^0) &\equiv D_\alpha \left(\hat{V}^{x^k} \frac{\partial \mathcal{L}^0}{\partial x^k_\alpha} + V^\alpha \mathcal{L}^0 \right) \\ &= \mathcal{L}^0 D_\alpha V^\alpha + V^\alpha D_\alpha \mathcal{L}^0 + \hat{V}^{x^k} \left[D_\alpha \left(\frac{\partial \mathcal{L}^0}{\partial x^k_\alpha} \right) - \frac{\partial \mathcal{L}^0}{\partial x^k} \right] \\ &\quad + \hat{V}^{x^k} \frac{\partial \mathcal{L}^0}{\partial x^k} + D_\alpha\left(\hat{V}^{x^k}\right) \frac{\partial \mathcal{L}^0}{\partial x^k_\alpha} \\ &= \mathcal{L}^0 D_\alpha V^\alpha + V^\alpha D_\alpha \mathcal{L}^0 - \hat{V}^{x^k} E_{x^k}(\mathcal{L}^0) + \hat{X} \mathcal{L}^0. \end{aligned} \tag{B.7}$$

However, $\tilde{X} = \hat{X} + V^\alpha D_\alpha$ from (3.11). Hence (B.7) reduces to

$$D_\alpha(W^\alpha + V^\alpha \mathcal{L}^0) = \tilde{X} \mathcal{L}^0 + \mathcal{L}^0 D_\alpha V^\alpha - \hat{V}^{x^k} E_{x^k}(\mathcal{L}^0), \tag{B.8}$$

which is equivalent to (3.13).

Appendix C

In this appendix we consider 2D MHD, with an ignorable coordinate z in two Cartesian space dimensions, in which the magnetic field $\mathbf{B} = (0, 0, B)^t$ lies along the z axis, and $B = B(x, y, t)$, and the adiabatic index of the gas is taken as $\gamma = 2$. The Lie symmetries and Lie algebra of the MHD equations in this case were investigated by Fuchs and Richter (1987). In this appendix we consider the conservation law associated with the projective symmetry, X_{14}, which has the form

$$X_{14} = t x^\mu \frac{\partial}{\partial x^\mu} + (x^i - u^i t) \frac{\partial}{\partial u^i} - 2t\rho \frac{\partial}{\partial \rho} - 4tp \frac{\partial}{\partial p} - 2tB \frac{\partial}{\partial B}. \tag{C.1}$$

Note that $x_{ij} \equiv \partial x^i/\partial x_0^j$ in the requirement that $\mathbf{B} = (0, 0, B)^t$, and the frozen in field theorem (2.9) implies that $x_{13} = x_{31} = x_{23} = x_{32} = 0$ and $x_{33} = 1$. The Lagrangian densities $\mathcal{L}$ and $\mathcal{L}^0$ are given by

$$\mathcal{L}^0 = \frac{1}{2}\rho_0|\mathbf{x}_t|^2 - \left(p_0 + \frac{B_0^2}{2\mu}\right)\frac{1}{J}, \qquad \mathcal{L} = \frac{\mathcal{L}^0}{J} = \frac{1}{2}\rho|\mathbf{u}|^2 - \left(p + \frac{B^2}{2\mu}\right). \tag{C.2}$$

Note in (C.2) that $B = B_0/J$ and $p = p_0/J^2$, where $J = \det(x_{ij})$.

Condition (3.9) for the symmetry to be a divergence symmetry of the action, for the case $V^{\mathbf{x}_0} = 0$ reduces to (6.13), which is of the same form as that satisfied by $\tilde{X}_{14}$ in the gas dynamical case. From (3.4)–(3.7) it follows that the density I^0 and flux components $\{I^j\}$ in the Lagrangian conservation law (3.4) are given by

$$I^0 = t(\mathbf{x} - \mathbf{u}t)\cdot\rho_0\mathbf{u} + t^2\mathcal{L}^0 - \frac{1}{2}\rho_0|\mathbf{x}|^2, \tag{C.3}$$

$$I^j = t(x^k - u^k t)\left[\left(p + \frac{B^2}{2\mu}\right)A_{kj} - \frac{B^2}{\mu}\delta^{ks}A_{3j}\right], \tag{C.4}$$

where $A_{kj} = \mathrm{cofac}(x_{kj}$. Using (3.16) *et seq*, the corresponding Eulerian conservation law has conserved density F^0 and flux $\mathbf{F}$ given by

$$F_0 = t\,(\mathbf{x} - \mathbf{u}t)\cdot\rho\mathbf{u} + t^2\mathcal{L} - \frac{1}{2}\rho|\mathbf{x}|^2, \tag{C.5}$$

$$\mathbf{F} = t\,(\mathbf{x} - \mathbf{u}t)\cdot\left[\rho\mathbf{u}\mathbf{u} + \left(p + \frac{B^2}{2\mu}\mathbf{I}\right) - \frac{\mathbf{B}\mathbf{B}}{\mu}\right] + \mathbf{u}\left(t^2\mathcal{L} - \frac{1}{2}\rho|\mathbf{x}|^2\right). \tag{C.6}$$

For completeness, we note that Fuchs and Richter (1987) obtained conservation laws associated with the projective symmetry (C.1) by using the method of Ibragimov (1985) alluded to in (6.17) *et seq*.

Appendix D

In this appendix we discuss the Jacobi identity,

$$[[X_a, X_b], X_c] + [[X_b, X_c], X_a] + [[X_c, X_a], X_b] = 0, \tag{D.1}$$

for vector fields $\{X_a, X_b, X_c\}$ for the fluid relabelling symmetries described by (7.1)–(7.9). The Jacobi identity is easily proved by using the Lie bracket definition (7.10) to expand the left-hand side of (D.1) as a sum of operators of the form $X_aX_bX_c$ and collecting like terms. However, it is also interesting to show the detailed dependence of the Jacobi identity on the functions Ω_a, Ω_b and Ω_c by using (7.8) and (7.9). Using the notation $X_{abc} = [[X_a, X_b], X_c]$, the Jacobi identity (D.1) is equivalent to

$$S_{abc} = X_{abc} + X_{bca} + X_{cab} \equiv -\left(V_{abc}^{\mathbf{x}_0} + V_{bca}^{\mathbf{x}_0} + V_{cab}^{\mathbf{x}_0}\right)\cdot\nabla_0\mathbf{x} = 0, \tag{D.2}$$

where

$$V_{abc}^{\mathbf{x}_0} = \frac{\nabla_0 S\times\nabla_0\Omega_{abc}}{\rho_0}, \qquad \Omega_{abc} = \frac{\nabla_0\Omega_{ab}\times\nabla_0\Omega_c\cdot\nabla_0 S}{\rho_0},$$
$$\Omega_{ab} = \frac{\nabla_0\Omega_a\times\nabla_0\Omega_b\cdot\nabla_0 S}{\rho_0}. \tag{D.3}$$

Using notations (D.2) and (D.3), the Jacobi identity is equivalent to the equation

$$\mathbf{T} = \nabla_0 S\times\nabla_0\left(\Omega_{abc} + \Omega_{bca} + \Omega_{cab}\right) = 0. \tag{D.4}$$

Noting that

$$\Omega_{abc} \equiv \frac{\nabla_0 S \cdot \nabla_0 \times (\Omega_{ab} \nabla_0 \Omega_c)}{\rho_0}, \tag{D.5}$$

we find

$$\Sigma = \Omega_{abc} + \Omega_{bca} + \Omega_{cab} = \frac{\nabla_0 S \cdot \nabla_0 \times \mathbf{W}}{\rho_0}, \tag{D.6}$$

where

$$\mathbf{W} = \Omega_{ab} \nabla_0 \Omega_c + \Omega_{bc} \nabla_0 \Omega_a + \Omega_{ca} \nabla_0 \Omega_b. \tag{D.7}$$

If

$$K = \nabla_0 \Omega_a \times \nabla_0 \Omega_b \cdot \nabla_0 \Omega_c \equiv \frac{\partial(\Omega_a, \Omega_b, \Omega_c)}{\partial(x_0, y_0, z_0)} \neq 0, \tag{D.8}$$

then we can use $(\Omega_a, \Omega_b, \Omega_c)$ as independent variables to replace (x_0, y_0, z_0). Using the above transformation of independent variables we find

$$\Omega_{ab} = \frac{K}{\rho_0} \frac{\partial S}{\partial \Omega_c}, \qquad \Omega_{bc} = \frac{K}{\rho_0} \frac{\partial S}{\partial \Omega_a}, \qquad \Omega_{ca} = \frac{K}{\rho_0} \frac{\partial S}{\partial \Omega_b}, \tag{D.9}$$

where we have used the expansion

$$\nabla_0 S = \frac{\partial S}{\partial \Omega_a} \nabla_0 \Omega_a + \frac{\partial S}{\partial \Omega_b} \nabla_0 \Omega_b + \frac{\partial S}{\partial \Omega_c} \nabla_0 \Omega_c. \tag{D.10}$$

Using (D.9) and (D.10) in (D.7) we obtain

$$\mathbf{W} = \frac{K}{\rho_0} \nabla_0 S. \tag{D.11}$$

Substitution of (D.11) in (D.6) gives

$$\Sigma = \frac{\nabla_0 S}{\rho_0} \cdot \nabla_0 \times \left(\frac{K}{\rho_0} \nabla_0 S \right) = 0. \tag{D.12}$$

Since $\Sigma = 0$ then $\mathbf{T} = 0$ in (D.4). This proves the Jacobi identity for this case.

If $K = 0$ then $\Omega_c = f(\Omega_a, \Omega_b)$, for some function f because Ω_a, Ω_b and Ω_c are dependent functions if $K = 0$. A straightforward calculation of $\mathbf{W}$ in this case shows that $\mathbf{W} = 0$, $\Sigma = 0$ and $\mathbf{T} = 0$ so that the Jacobi identity also holds in this case as well. The above analysis suggests that the geometry of vector fields X_j is more transparent if we use an independent set of the Ω_j as the independent variables.

References

Bluman G W and Kumei S 1989 *Symmetries and Differential Equations* (New York: Springer)

Bridges T J, Hydon P E and Reich S 2005 Vorticity and symplecticity in Lagrangian fluid dynamics *J. Phys. A: Math. Gen.* **38** 1403–18

Dewar R L 1970 Interaction between hydromagnetic waves and a time-dependent inhomogeneous medium *Phys. Fluids* **13** 2710–20

Fuchs J C 1991 Symmetry groups and similarity solutions of the MHD equations *J. Math. Phys.* **32** 1703–8

Fuchs J C and Richter E W 1987 Similarity solutions for the two-dimensional non-stationary ideal MHD eqautions *J. Phys. A: Math. Gen.* **20** 3135–57

Gibbon J D, Holm D D, Kerr R M and Roulstone I 2006 Quaternions and particle dynamics in the Euler fluid equations *Nonlinearity* **19** 1969–83

Grundland A M and Lalague L 1995 Lie subgroups of symmetry groups of fluid dynamics and magnetohydrodynamics equations *Can. J. Phys.* **73** 463–77

Harrison B K and Estabrook F B 1971 Geometric approach to invariance groups and solution of partial differential systems *J. Math. Phys.* **12** 653–66

Holm D D 1999 Fluctuation effects on 3D Lagrangian mean and Eulerian mean fluid motion *Physica* D **133** 215–69

Holm D D and Kupershmidt B A 1983a Poisson brackets and Clebsch representations for magnetohydrodynamics, multi-fluid plasmas, and elasticity *Physics* D **6D** 347–63

Holm D D and Kupershmidt B A 1983b Noncanonical Hamiltonian formulation of ideal magnetohydrodynamics *Physica* D **7** 330–3

Holm D D, Marsden J E and Ratiu T S 1998 The Euler–Lagrange equations and semiproducts with application to continuum theories *Adv. Math.* **137** 1–81

Hydon P E 2005 Multisymplectic conservation laws for differential and differential-difference equations *Proc. R. Soc.* A **461** 1627–37

Ibragimov N H 1985 *Transformation Groups Applied to Mathematical Physics* (Dordrecht: Reidel)

Ibragimov N H 1994 *Lie Group Analysis of Differential Equations: Symmetries, Exact Solutions and Conservation Laws* vol 1 ed N H Ibragimov (Boca Raton, FL: CRC Press)

Kuznetsov E A and Ruban V P 2000 Hamiltonian dynamics of vortex and magnetic lines in hydrodynamic type systems *Phys. Rev.* E **61** 831–41

Lundgren T S 1963 Hamilton's variational principle for a perfectly conducting plasma continuum *Phys. Fluids* **6** 898–904

Marsden J E and Ratiu T S 1994 *Introduction to Mechanics and Symmetry (Texts in Appl. Math. vol 17)* (New York: Springer)

Morrison P J 1982 Poisson brackets for fluids and plasmas *Mathematical Methods in Hydrodynamics and Integrability of Dynamical Systems (AIP Proc. Conf. vol 88)* ed M Tabor and Y M Treve pp 13–46

Morrison P J 1998 Hamiltonian description of the ideal fluid *Rev. Mod. Phys.* **70** 467–521

Newcomb W A 1962 Lagrangian and Hamiltonian methods in magnetohydrodynamics *Nucl. Fusion Suppl.* Part 2 451–63

Nutku Y 1987 On a new class of completely integrable nonlinear wave equations II. Multi-Hamiltonian structure *J. Math. Phys.* **28** 2579–85

Olver P J 1993 *Applications of Lie Groups to Differential Equations* 2nd edn (New York: Springer)

Olver P J and Nutku Y 1988 Hamiltonian structures for systems of hyperbolic conservation laws *J. Math. Phys.* **29** 1610–9

Olver P J and Pohjanpelto J 2005 Maurer-Cartan forms and the structure of Lie pseudo-groups *Selecta Math.* **11** 99–126

Ovsjannikov L V 1962 Group Properties of Differential Equations, Translated by G.W. Bluman, available from Caltech., translation from the Russian version, Gruppovye Svoysta Differentisialny Uravnenii, Novosibirsk, Siberian section of the Academy of Sciences U.S.S.R

Padhye N S 1998 Topics in Lagrangian and Hamiltonian Fluid Dynamics: Relabeling Symmetry and Ion Acoustic Wave Stability *Ph D Dissertation* University of Texas at Austin

Padhye N and Morrison P J 1996a Fluid element relabeling symmetry *Phys. Lett.* A **219** 287–92

Padhye N and Morrison P J 1996b Relabeling symmetries in hydrodynamics and magnetohydrodynamics *Plasma Phys. Reports* **22** 869–77

Parker E N 1979 *Cosmical Magnetic Fields, Their Origin and Their Activity* (Oxford: Clarendon Press)

Salmon R 1982 Hamilton's principle and Ertel's theorem *Am. Inst. Phys. Conf. Proc.* **88** 127–35

Salmon R 1988 Hamiltonian fluid mechanics *Annu. Rev. Fluid Mech.* **20** 225–56

Stern D P 1966 The motion of magnetic field lines *Space Sci. Rev.* **6** 147–73

Webb G M, Zank G P, Kaghashvili E Kh and Ratkiewicz R E 2005a Magnetohydrodynamic waves in non-uniform flows, I: a variational approach *J. Plasma Phys.* **71** 785–809

Webb G M, Zank G P, Kaghashvili E Kh and Ratkiewicz R E 2005b Magnetohydrodynamic waves in non-uniform flows, II: stress-energy tensors, conservation laws and Lie symmetries *J. Plasma Phys.* **71** 811–57

Zakharov V E and Kuznetsov E A 1984 Hamiltonian formalism for systems of hydrodynamic type *Soviet Scientific Reviews, Section C: Mathematical Physics Reviews* vol 4 ed S P Novikov (Chur: Harwood Academic) pp 167–220